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Risk minimization and portfolio diversification

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1. Introduction

It is considered 'common sense' among financial investors to maximize the portfolio return while satisfying some risk constraint. The mean-variance technique addressing this problem has been introduced by Markowitz in 1952 and is developed in Markowitz (1991). The objective in portfolio selection is decreasing the investment downside risk; this risk is quantified through various measures like value at risk (VaR). These notions of risk measures in portfolio selection and risk management have resulted in a great deal of published literature. For example, the notion of VaR, as the α-quantile subtracted from the *mean* of the portfolio return has been thoroughly investigated in Duffie and Pan (1997) and Jorion (2007), which turns out to suffer from being a non-coherent risk measure. Capital at risk (CaR) is introduced to resolve this problem. CaR differs from VaR by a constant (it is VaR adjusted to the riskless return). Some other risk measures, such as average value at risk (AVaR) and limited expected loss, were introduced to address the shortcomings of VaR. Analytical formulas for these types of risk measures, as well as risk constrained portfolio optimization in a continuous time framework are provided in Gambrah and Pirvu (2014).

Portfolio selection under bounded CaR is well explored in Emmer *et al.* (2001). In the Black–Scholes setting with constant coefficients, they obtained a closed form solution for the optimal portfolio with maximum mean return, subject to a bounded CaR. It has been shown in Emmer *et al.* (2001) that the use of merely variance-based measures leads to a decreased proportion of risky assets in a portfolio when

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the planning horizon increases, which can be resolved when CaR is employed; this argument supports the use of CaR in risk management. The results in Emmer *et al.* (2001) have been extended to allow the no-short-selling constraint in Dmitrašinović-Vidović *et al.* (2011). The counter-intuitive behaviour in VaR constrained optimized portfolios, with an increase in investment time horizon has been shown in Dmitrašinović-Vidović and Ware (2006).

The literature on portfolio selection given a correlation constraint is rather limited. Bernard and Vanduffel (2014) studied mean variance optimal portfolios in the presence of a stochastic benchmark correlated with the market, and discussed how their method could be used to detect fraud in financial reports. For example, under some conditions one could not have a positive Sharpe ratio while having a negative correlation with the market index. Bernard *et al.* (2015) investigated optimal portfolio selection with state-dependent preferences, and the optimal portfolio of this paper in a complete market can also be derived by following their approach. However, we mainly consider an incomplete market model and the complete market case is obtained as a special case of the result under an incomplete market model.

In an incomplete market Black–Scholes setting, this paper provides the closed form solution to the CaR minimizing portfolio that satisfies a correlation constraint between the investor's terminal wealth and a given index process. One possible choice of this index is the growth optimal portfolio (GOP) according to Bernard and Vanduffel (2014). Applying the correlation constraint is useful in some situations. For example, maintaining a negative correlation with the market index allows one to better control the risky investment during a market crash (i.e. when the market index is heavily declining); in these situations, the negative correlation could rescue the portfolio from collapse. By analysing the closed form solutions of the constrained and unconstrained portfolio selection problems, we notice that the correlation constraint leads to more diversified portfolios if variance is used as the measure of diversification.

The rest of the paper is organized as follows. In section 2, we introduce the market model, and our specific notion of CaR. Section 3 briefly reiterates the minimization problem of CaR. Section 4 is devoted to the problem of CaR minimization under the correlation constraint. The complete market model is considered in section 5.1, which is followed by some numerical examples. The paper concludes in section 6. The proofs of the results are delegated to appendix 1.

2. Model

Consider a probability space $(\Omega, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, which accommodates a standard multidimensional Brownian motion. Throughout this paper, we restrict ourselves to a geometric Brownian motion model for stock prices. Let us consider a financial market model with the following specifications.

- Assets are traded continuously over a finite time horizon $[0, T]$ in a frictionless market.
- There is one risk-free asset, denoted by $S_0(t)$, with positive constant interest rate r:
$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r\mathrm{d}t.$$
- There are m risky assets (stocks), driven by a d-dimensional Brownian motion $\mathbf{W}(t) = [W_1(t)\ldots W_d(t)]'$:
$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = (r+b_i)\mathrm{d}t + \sum_{j=1}^{d}\sigma_{ij}dW_j(t),$$
$$S_i(0) = s_i, \quad i = 1, \ldots, m.$$
The market can generally be *incomplete*, namely the number of assets m might be less than the dimension of Brownian motion ($m \le d$). Here $\sigma = [\sigma_{ij}]$ is the $m \times d$ volatility matrix, such that $\sigma\sigma'$ is invertible. This condition rules out the market arbitrage. Moreover, $\mathbf{b} = \left[b_1 \ldots b_m\right]'$ is the vector of excess return rate of each risky asset, that we take it to be positive.
- Let $\boldsymbol{\pi} = \left[\pi_1 \ldots \pi_m\right]' \in \mathbb{R}^m$ be the portfolio vector of the investor, where π_i indicates the fraction of the total initial wealth x invested in stock i, that is assumed to be constant over time. Therefore, $\pi_0 = 1 - \mathbf{1}'\boldsymbol{\pi}$ is the fraction of wealth invested in the risk-free asset. We draw attention to the constant proportions, which are time invariant, and more tractable. The constant $\boldsymbol{\pi}$ does not mean there is no trade, since one needs to rebalance the portfolio continuously to keep the portion invested in each asset unchanged over time.
The stochastic differential equation of the wealth process is
$$\frac{\mathrm{d}X^{\pi}(t)}{X^{\pi}(t)} = (r+\mathbf{b}'\boldsymbol{\pi})\mathrm{d}t + \boldsymbol{\pi}'\sigma d\mathbf{W}(t), \quad X^{\pi}(0) = x.$$
Then direct computations lead to
$$X^{\pi}(T) = xe^{[(r+\mathbf{b}'\boldsymbol{\pi}-\|\sigma'\boldsymbol{\pi}\|^2/2)T+\boldsymbol{\pi}'\sigma\mathbf{W}(T)]},$$
$$\mathbb{E}\left(X^{\pi}(T)\right) = xe^{(r+\mathbf{b}'\boldsymbol{\pi})T},$$
$$\mathrm{Var}(X^{\pi}(T)) = x^2e^{2(r+\mathbf{b}'\boldsymbol{\pi})T}\left(e^{\|\sigma'\boldsymbol{\pi}\|^2T}-1\right),$$
$$\mathrm{Var}(\log X^{\pi}(T)) = T\boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\pi}.$$

In order to preserve the tractability, risk measurements are performed for *logarithmic* returns, rather than arithmetic returns. It is well known that for small time horizons, the two types of returns are close to each other. Keeping this notion in mind, we can now present a formal definition of CaR.

Definition 1 (Capital at Risk) Let z_α be the α-quantile of the standard Gaussian distribution. The CaR of a fixed portfolio vector $\boldsymbol{\pi}$ is defined as the difference of the riskless Log return and the α-quantile of the Log return over $[0, T]$:
$$\begin{aligned}
q(x,\boldsymbol{\pi},\alpha,T) &= \inf\{z\in\mathbb{R} : \mathbb{P}\left(\log\left(\frac{X^{\pi}(T)}{X^{\pi}(0)}\right)\le z\right)\ge\alpha\}\\
&= (r+\mathbf{b}'\boldsymbol{\pi})T - \frac{1}{2}\left\|\sigma'\boldsymbol{\pi}\right\|^2T + z_\alpha\left\|\sigma'\boldsymbol{\pi}\right\|\sqrt{T}\\
\mathrm{CaR}(\boldsymbol{\pi},\alpha,T) &:= rT - q(x,\boldsymbol{\pi},\alpha,T)\\
&= -\mathbf{b}'\boldsymbol{\pi}T + \frac{1}{2}\left\|\sigma'\boldsymbol{\pi}\right\|^2T - z_\alpha\left\|\sigma'\boldsymbol{\pi}\right\|\sqrt{T}.
\end{aligned}$$
We assume that $\alpha < 0.5$, which means $z_\alpha < 0$.

3. CaR minimization

CaR minimization is first explored in Emmer *et al*. (2001), in which a closed form solution is found for the portfolio with maximum expected return under a bounded CaR constraint. However, the analysis is for the *complete* market. The next proposition studies the CaR minimization under incomplete market and logarithmic return. A similar problem, but for arithmetic return under complete market hypothesis, has been studied in Emmer *et al*. (2001).

PROPOSITION 1 *The minimum CaR portfolio, i.e. the solution of:*

$$\operatorname*{argmin}_{\boldsymbol{\pi}\in\mathbb{R}^d} \mathrm{CaR}(\boldsymbol{\pi},\alpha,T)$$

satisfies

$$\sigma'\boldsymbol{\pi}^* = \left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|\right)^+ \frac{\sigma'(\sigma\sigma')^{-1}\mathbf{b}}{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|}. \quad (1)$$

Moreover, the minimum CaR is

$$\mathrm{CaR}(\boldsymbol{\pi}^*,\alpha,T) = -\frac{T}{2}\left[\left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|\right)^+\right]^2. \quad (2)$$

The proof of this proposition is given in the appendix 1. As a straightforward corollary of this proposition, one can find the CaR minimizing portfolio in the complete market case (the volatility matrix σ is invertible). Indeed, in this case $\boldsymbol{\pi}^*$ is obtained from (1) to be:

$$\boldsymbol{\pi}^* = \left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma^{-1}\mathbf{b}\right\|\right)^+ \frac{(\sigma\sigma')^{-1}\mathbf{b}}{\left\|\sigma^{-1}\mathbf{b}\right\|}. \quad (3)$$

4. CaR minimization under correlation constraint

In this section, we focus on minimizing the CaR subject to a correlation constraint. In other words, we want to find the optimal portfolio that minimizes the CaR, as well as satisfies a correlation constraint with another index process. Assume that the index dynamics is given by:

$$\frac{\mathrm{d}Y(t)}{Y(t)} = (r + \mathbf{b}'\boldsymbol{\eta})\mathrm{d}t + \boldsymbol{\eta}'\sigma d\mathbf{W}(t), \qquad Y(0) = y, \quad (4)$$

where η is the index portfolio. Moreover, we assume that the target process has positive excess return over r, i.e. $\mathbf{b}'\boldsymbol{\eta} > 0$, and enforce that the correlation between the log values of $X(T)$ and $Y(T)$ does not exceed a negative threshold. This condition is expressed as:

$$\mathrm{Corr}(\log X(T), \log Y(T)) \le -\delta, \quad \text{where} \quad \delta \ge 0.$$

The Y process can be any financial index or wealth process, which is driven by the same sources of uncertainty as stocks in a market. The correlation between terminal log values is found in a closed form as follows.

$$\mathrm{Corr}(\log X(T), \log Y(T)) = \frac{\boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\eta}}{\|\sigma'\boldsymbol{\pi}\|\,\|\sigma'\boldsymbol{\eta}\|}.$$

Then the risk minimization problem under correlation constraint is:

$$\begin{aligned} &\min_{\boldsymbol{\pi}\in\mathbb{R}^m} \ \mathrm{CaR}(\boldsymbol{\pi},\alpha,T) \\ &\text{subject to } \ \mathrm{Corr}(\log X(T), \log Y(T)) \le -\delta. \end{aligned} \quad (5)$$

Notice that we consider the following problem, rather than the problem in (5).

$$\begin{aligned} &\min_{\boldsymbol{\pi}\in\mathbb{R}^m} \left(-\mathbf{b}'\boldsymbol{\pi}T + \frac{1}{2}\left\|\sigma'\boldsymbol{\pi}\right\|^2 T - z_\alpha\left\|\sigma'\boldsymbol{\pi}\right\|\sqrt{T}\right) \\ &\text{subject to } \ \delta\left\|\sigma'\boldsymbol{\eta}\right\|\left\|\sigma'\boldsymbol{\pi}\right\| + \boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\eta} \le 0. \end{aligned} \quad (6)$$

There is a subtle difference between (5) and (6), which arises for zero portfolio $\boldsymbol{\pi}$. This vector is contained in the region of the second optimization problem, but not in the first, because the correlation cannot be defined for the zero portfolio. Hence, it is not surprising if the zero vector happens to be the optimal portfolio of problem (6). The condition of negative correlation is imposed for tractability reasons (it renders the optimization problem convex). Moreover, it has a financial interpretation: in times of financial downturns, it is desirable to be negatively correlated with the market index $Y(t)$. Because of the explicit formulation of the problem, the solution can be found analytically with convex optimization methods. The following theorem presents the main result of the paper.

THEOREM 1 *The optimal portfolios that solve* (6) *should satisfy: The parameter* λ^* *is given by*

$$\sigma'\boldsymbol{\pi}^* = \sigma'\left[\frac{\sqrt{1-\delta^2}\left(\frac{z_\alpha\|\sigma'\boldsymbol{\eta}\|}{\sqrt{T}} + \sqrt{1-\delta^2}\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\|\sigma'\boldsymbol{\eta}\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2} - \delta\mathbf{b}'\boldsymbol{\eta}\right)^+\left((\sigma\sigma')^{-1}\mathbf{b}T - \lambda^*\boldsymbol{\eta}\right)}{T\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\|\sigma'\boldsymbol{\eta}\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2}}\right]. \quad (7)$$

$$\lambda^* = \frac{1}{\|\sigma'\boldsymbol{\eta}\|^2} \times \left(\mathbf{b}'\boldsymbol{\eta}T + \frac{T\delta}{\sqrt{1-\delta^2}}\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\|\sigma'\boldsymbol{\eta}\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2}\right),$$

and the minimum CaR is

$$CaR(\boldsymbol{\pi}^*,\alpha,T) = \frac{-T}{2\|\sigma'\boldsymbol{\eta}\|^2}\left[\left(\frac{z_\alpha\left\|\sigma'\boldsymbol{\eta}\right\|}{\sqrt{T}} + \sqrt{1-\delta^2}\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\|\sigma'\boldsymbol{\eta}\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2} - \delta\mathbf{b}'\boldsymbol{\eta}\right)^+\right]^2. \quad (8)$$

The proof of theorem 1 is given in the appendix 1. Similar to the unconstrained case in section 3, the complete market case of constrained problem can be obtained by assuming that σ is an invertible square matrix. Then, simplified versions of the optimal portfolio and the minimal CaR, which correspond to (7) and (8), respectively, can be derived. It is worth mentioning that since the system in (7) is underdetermined if the market is incomplete, there are infinitely many portfolio vectors that satisfy (7) and have same CaR value. However, the optimal portfolio is uniquely determined if the market is complete, i.e. when σ is invertible.

Remark 1 In complete market, our result can be obtained as a special case of Bernard *et al*. (2015), and the optimal portfolio of complete market can also be derived by using the techniques in Bernard *et al*. (2015). However, our main result is obtained in incomplete market, which is not considered in Bernard *et al*. (2015); and furthermore our complete market case is obtained as a corollary of the results in an incomplete market.

Remark 2 The optimal portfolio in (7) has a two-fund separation structure: the first component is similar to the solution of unconstrained case in (1); and the second component is induced by the correlation constraint. In the constrained case, unless the zero portfolio is optimal, the correlation constraint always binds, that is, $\lambda^* > 0$. This is because the mean excess return of index process is assumed to be positive, i.e. $\mathbf{b}'\boldsymbol{\eta} > 0$. Since the mean excess return of index process is positive, it is optimal to have greatest correlation with index process under the correlation constraint, which is $-\delta$. More details can be ascertained from the proof of theorem 1.

Example 1 The main purpose of this example is to identify the effect of the correlation constraint in risk reduction under an incomplete market situation. We take the simple scenario of two stocks with the given volatility matrix, and excess return rate vector:

$$\sigma := \begin{bmatrix} \sigma_1 & 0 & \rho \\ 0 & \sigma_2 & 0 \end{bmatrix} \quad \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The positive volatility ρ is the idiosyncratic risk of the first asset. Let us consider the case where the benchmark portfolio contains the second stock, i.e. $\boldsymbol{\eta} := \begin{bmatrix} 0 & \eta_2 \end{bmatrix}'$. If we denote the optimal portfolio of the unconstrained problem as $\boldsymbol{\pi}^*$ and that of the constrained problem as $\boldsymbol{\pi}_c^*$, then we can easily compute the variances of their log returns as follows:

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{\frac{b_1^2}{\sigma_1^2+\rho^2} + \frac{b_2^2}{\sigma_2^2}}\right)^+\right]^2,$$

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}_c^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{1-\delta^2}\frac{b_1}{\sqrt{\sigma_1^2+\rho^2}} - \delta b_2\eta_2\right)^+\right]^2.$$

Since $\mathbf{b}'\boldsymbol{\eta} = b_2\eta_2 > 0$, the variance in the constrained problem becomes lower, thus yielding a more diversified portfolio.

5. The complete market case

5.1. Choice of benchmark process in complete market

In this section, we study the special choice of a benchmark process as the target index. Moreover, we assume that the market is complete, that is, the number of available stocks m is identical to the Brownian motion dimension d and σ is invertible. The assets are regrouped into two parts: the risky assets that are included in the index portfolio η, and some additional risky assets that are available to the investor's portfolio π, but not a part of the index portfolio η. Let us decompose the Brownian motion vector as $\mathbf{W}(t) = \begin{bmatrix}\mathbf{W}_1(t)' & \mathbf{W}_2(t)'\end{bmatrix}'$, where the first component is ℓ dimensional. Moreover, without any loss of generality, we can represent the volatility matrix and its inverse as:

$$\sigma = \begin{bmatrix} \sigma_{11} & \mathbf{0} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad \sigma^{-1} = \begin{bmatrix} \sigma_{11}^{-1} & \mathbf{0} \\ -\sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1} & \sigma_{22}^{-1} \end{bmatrix}. \tag{9}$$

Here σ_{11} and σ_{22} are square ℓ and $d-\ell$ matrices, respectively. And σ_{21} and $\mathbf{0}$ are $(d-\ell)\times\ell$ and $\ell\times(d-\ell)$ matrices, respectively. Consequently, there are two types of stocks: the first type is only driven by $\mathbf{W}_1(t)$, and the second type is driven by both components of the Brownian motion. Following the similar setting to the previous section, the investor's wealth process is expressed as

$$\begin{aligned}\frac{\mathrm{d}X(t)}{X(t)} &= (r+\mathbf{b}'\boldsymbol{\pi})\mathrm{d}t + \begin{bmatrix}\boldsymbol{\pi}_1' & \boldsymbol{\pi}_2'\end{bmatrix}\begin{bmatrix} \sigma_{11} & \mathbf{0} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} d\mathbf{W}_1(t) \\ d\mathbf{W}_2(t) \end{bmatrix} \\ &= (r+\mathbf{b}'\boldsymbol{\pi})\mathrm{d}t + (\boldsymbol{\pi}_1'\sigma_{11} + \boldsymbol{\pi}_2'\sigma_{21})d\mathbf{W}_1(t) \\ &\quad + \boldsymbol{\pi}_2'\sigma_{22}d\mathbf{W}_2(t),\end{aligned}$$

where $\boldsymbol{\pi}' = \begin{bmatrix}\boldsymbol{\pi}_1' & \boldsymbol{\pi}_2'\end{bmatrix}$ and $\mathbf{b}' = \begin{bmatrix}\mathbf{b}_1' & \mathbf{b}_2'\end{bmatrix}$. $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are portfolio vectors for the first type assets and the second type assets, respectively, whereas $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are average excess return vectors for the first type assets and the second type assets, respectively. Note that $\boldsymbol{\pi}_1$ and $\boldsymbol{b}_1$ are l dimensional column vectors, and $\boldsymbol{\pi}_2$ and $\boldsymbol{b}_1$ are $d-l$ dimensional column vectors.

Following Bernard and Vanduffel (2014), the index process is taken to be the GOP. As argued in Platen (2006), under some conditions GOP is the inverse of stochastic discount factor, or equivalently the numéraire process, as in Christensen and Larsen (2007). Let us re-emphasize that only the first type of stocks is considered to construct the market index. Hence, if $\theta = \sigma_{11}^{-1}\mathbf{b}_1$ denotes the market price of risk of the first type of stocks, the stochastic discount factor ξ is governed by:

$$\frac{\mathrm{d}\xi(t)}{\xi(t)} = -r\mathrm{d}t - \theta' d\mathbf{W}_1(t).$$

Consequently, the index process $Y(t) = \xi(t)^{-1}$ satisfies the following SDE:

$$\begin{aligned}\frac{\mathrm{d}Y(t)}{Y(t)} &= (r+\|\theta\|^2)\mathrm{d}t + \theta' d\mathbf{W}_1(t) \\ &= \left(r + \left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|^2\right)\mathrm{d}t + ((\sigma_{11}\sigma_{11}')^{-1}\mathbf{b}_1)'\sigma_{11}d\mathbf{W}_1(t).\end{aligned}$$

One can compare (4) with the dynamics of GOP to take the index portfolio $\boldsymbol{\eta}$ as $\begin{bmatrix}((\sigma_{11}\sigma_{11}')^{-1}\mathbf{b}_1)' & \mathbf{0}\end{bmatrix}'$. The next proposition uses this special choice of index portfolio vector in theorem 1.

Proposition 2 *The optimal portfolio minimizing the CaR, and satisfies the correlation constraint is:*

$$\begin{aligned}\boldsymbol{\pi}^* &= \frac{\sqrt{1-\delta^2}\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{1-\delta^2}\left\|\sigma_{22}^{-1}\mathbf{b}_2 - \sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1}\mathbf{b}_1\right\| - \delta\left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|\right)^+}{T\left\|\sigma_{22}^{-1}\mathbf{b}_2 - \sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1}\mathbf{b}_1\right\|} \\ &\quad \times \left((\sigma\sigma')^{-1}\mathbf{b}T - \lambda^*\boldsymbol{\eta}\right),\end{aligned} \tag{10}$$

where λ^* *is given by*

$$\lambda^* = T\left(1 + \frac{\delta}{\left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|\sqrt{1-\delta^2}}\left\|\sigma_{22}^{-1}\mathbf{b}_2 - \sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1}\mathbf{b}_1\right\|\right). \tag{11}$$

The proof of proposition 2 is provided in the appendix 1, which is an immediate consequence of theorem 1.

Since we have closed form expressions in proposition 2, we can easily examine the impact of the correlation constraint on the portfolio diversification. Likewise example 1, to distinguish between the unconstrained case and the constrained case, all the optimal variables for the problem *with* the correlation constraint are written with subscript 'c,' like $\boldsymbol{\pi}_c^*$. The following proposition shows that, if we take the variance of log return as a measure of portfolio diversity, a more diversified portfolio can be obtained in the presence of the correlation constraint. The proof of the following proposition is given in the appendix 1.

PROPOSITION 3 *The optimal portfolio of the constrained problem* $\boldsymbol{\pi}_c^*$ *is more diversified than the optimal portfolio of the unconstrained problem* $\boldsymbol{\pi}^*$. *That is to say:*

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}^*}(T)) \geq \mathrm{Var}(\log X^{\boldsymbol{\pi}_c^*}(T)).$$

5.2. Diversification and risk control over market downfalls

As an application of our portfolio optimization problem with the correlation constraint, we would like to explore the diversification during the period of the market collapse. In order to be able to track the downfalls in the market and for ease of exposition, we assume that the market is complete and the first type of stocks is driven only by a single Brownian motion. Then by letting large enough values of σ_{11} (which would happen during the time of a market crash), we consider the asymptotic composition of the optimal portfolios (constrained and unconstrained). As a direct result of (10), we get:

$$\lim_{\sigma_{11}\to\infty}\boldsymbol{\pi}^* = \left(\frac{z_\alpha}{\left\|\sigma_{22}^{-1}\mathbf{b}_2\right\|\sqrt{T}} + 1\right)^+ \begin{bmatrix} 0 \\ (\sigma_{22}\sigma_{22}')^{-1}\mathbf{b}_2 \end{bmatrix},$$

$$\lim_{\sigma_{11}\to\infty}\boldsymbol{\pi}_c^* = \sqrt{1-\delta^2}\left(\frac{z_\alpha}{\left\|\sigma_{22}^{-1}\mathbf{b}_2\right\|\sqrt{T}} + \sqrt{1-\delta^2}\right)^+ \times \begin{bmatrix} 0 \\ (\sigma_{22}\sigma_{22}')^{-1}\mathbf{b}_2 \end{bmatrix}.$$

Let us mention the zero investment in the first type of stocks in both constrained and unconstrained optimal portfolios, which is expected owing to the increase of σ_{11}. On the other hand, the correlation constraint lowers the investment in the second class of stocks. The diversification benefit of the correlation constraint can also be seen from considering the asymptotic variances of $\log X^{\boldsymbol{\pi}^*}(T)$ and $\log X^{\boldsymbol{\pi}_c^*}(T)$, which are computed as:

$$\lim_{\sigma_{11}\to\infty}\mathrm{Var}(\log X^{\boldsymbol{\pi}^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma_{22}^{-1}\mathbf{b}_2\right\|\right)^+\right]^2,$$

$$\lim_{\sigma_{11}\to\infty}\mathrm{Var}(\log X^{\boldsymbol{\pi}_c^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{1-\delta^2}\left\|\sigma_{22}^{-1}\mathbf{b}_2\right\|\right)^+\right]^2.$$

5.3. Numerical experiments

In this section, we consider some numerical examples to shed light on the portfolio diversification achieved by imposing a correlation constraint. In the numerical experiments we employ the market parameters of Dmitrašinović-Vidović *et al*. (2011). The market consists of three stocks; $S_1(t)$ is the first type stock while $S_2(t)$ and $S_3(t)$ are second type stocks. Instead of the volatility matrix σ, Dmitrašinović-Vidović *et al*. (2011) provide parameters for the 3×3 correlation matrix ρ and standard deviations γ_1, γ_2 and γ_3 of returns of the three stocks. Then the covariance matrix is $\Sigma = \Gamma\rho\Gamma$, where $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3)$, and the corresponding volatility matrix σ of the form in (9) that satisfies $\sigma\sigma' = \Sigma$ can be obtained by Cholesky decomposition. We assume that the standard deviations of second type stocks are given by constants $\gamma_2 = 0.25$ and $\gamma_3 = 0.3$.

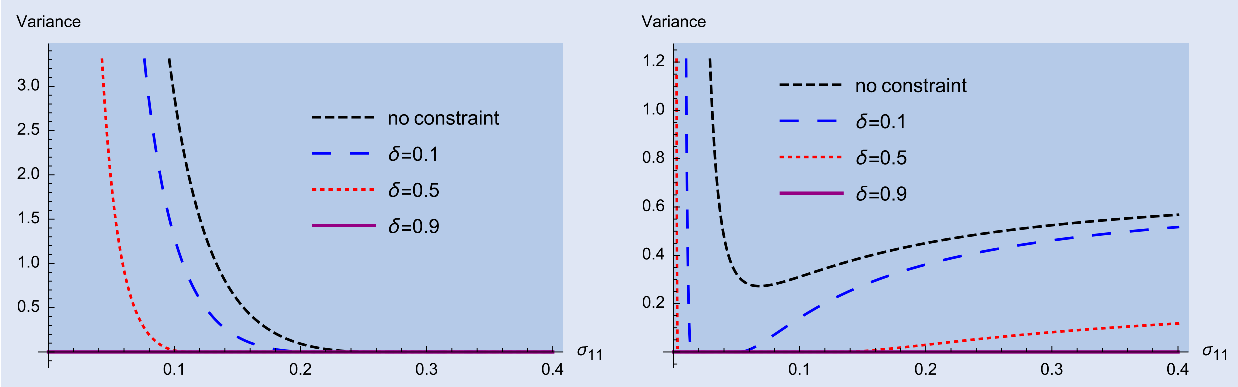
Example 2 Here we focus on the effect of market asset volatility on the variance of the log return. By increasing σ_{11} (equivalently, by increasing γ_1), we track the behaviour of log return variances (these are seen as measures of diversification). Two sets of correlation matrices and excess mean returns of stocks, as in Dmitrašinović-Vidović *et al*. (2011), are investigated:

$$\rho^{(1)} = \begin{bmatrix} 1.0 & -0.6 & -0.8 \\ -0.6 & 1.0 & 0.5 \\ -0.8 & 0.5 & 1.0 \end{bmatrix}, \quad b^{(1)} = \begin{bmatrix} 0.07 \; 0.05 \; 0.03 \end{bmatrix}', \tag{12}$$

$$\rho^{(2)} = \begin{bmatrix} 1.0 & -0.3 & 0.5 \\ -0.3 & 1.0 & -0.9 \\ 0.5 & -0.9 & 1.0 \end{bmatrix}, \quad b^{(2)} = \begin{bmatrix} 0.03 \; 0.05 \; 0.07 \end{bmatrix}'. \tag{13}$$

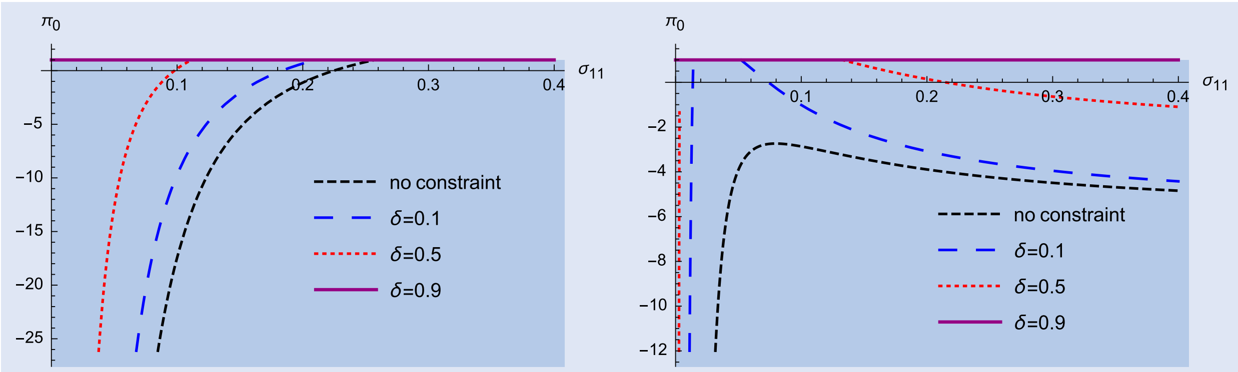
In figure 1, the plots of log return variances for these two sets of data are presented, wherein both the associated graphs for the constrained problem are depicted for three different values of the correlation threshold δ. Note that the highest curves in each plot corresponds to the unconstrained optimal portfolio. All the graphs are plotted for the fixed values of time horizon $T = 5$ and confidence level $\alpha = 0.05$. It is worthwhile to look at the effect of δ on the variance; higher values of δ lead to more diversification (lower variance). Extreme situations may happen: note that for $\delta = 0.9$ in figure 1(a) there is zero variance, which means pure risk-free investment.

Example 3 In this example, we illustrate how the fraction of investment on risk-free asset, i.e. π_0, is changing as a response to an increase in market volatility σ_{11}. The findings are presented for both data-sets of (12) and (13). Since we did not assume any restriction on borrowing/shortselling, negative values occur in some instances for optimal proportion of risk-free investment in both plots in figure 2. Let us note from both graphs that the bigger δ is, the higher risk-free investment would get (for $\delta = 0.9$ there is no investment on stocks and all portfolio is invested in the risk-free asset, which shows the reason why the log return has zero variance in this case). One should also observe the pattern of investing more on risk-free asset in figure 2(a) as a consequence of increase in market volatility, regardless of δ. However, this observation does not occur in figure 2(b), because of the structure of a stock correlation matrix.



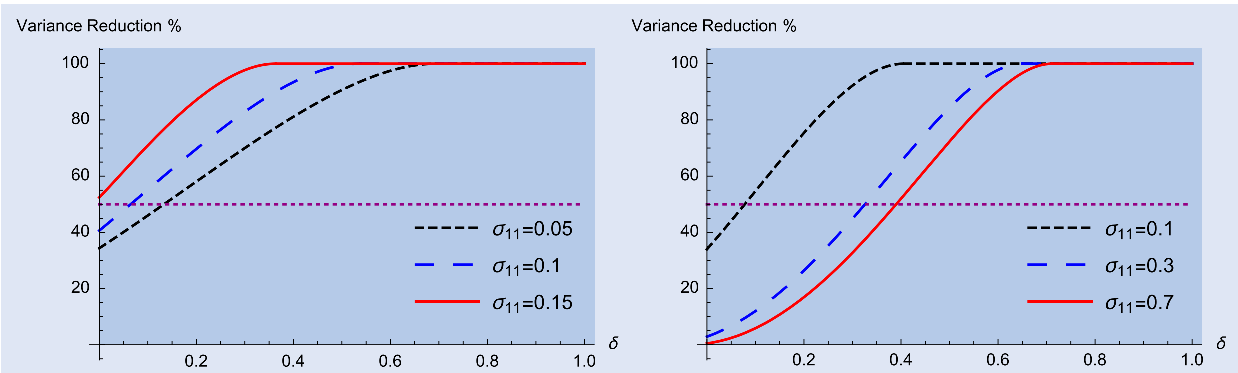
(a) variance for 1st set of data (b) variance for 2nd set of data

Figure 1. Log return variance.



(a) $\pi^*_{0,c}$ for 1st set of data (b) $\pi^*_{0,c}$ for 2nd set of data

Figure 2. Risk-free investment fraction.



(a) Variance reduction for 1st set of data (b) Variance reduction for 2nd set of data

Figure 3. Variance reduction percentage.

Example 4 In the two previous examples, it is illustrated that an increase in δ leads to a more diversified portfolio. In this example, we want to investigate this effect more precisely. Figure 3 shows the percentage of variance reduction from an unconstrained log return because of the correlation constraint. Both graphs show that by increasing δ, the reduction in variance increases. The dotted line draws the 50% variance reduction, which intercepts the curves at higher values of δ, as we consider more volatile cases in the second set of data.

6. Conclusion

In an incomplete market Black–Scholes setting with constant parameters, the optimal portfolios which minimize the CaR and achieve a negative prescribed correlation with a given financial index, are characterized analytically. It is shown that, under a certain choice of the financial market, the correlation constraint leads to a more diversified portfolio, that is, the variance of constrained optimal portfolios is lower than the variance of optimal unconstrained portfolios. Moreover, it turns out that the correlation constraint reduces the variance and increases the risk-free investment during market declines. Numerical experiments explore the effect on the optimal portfolio composition induced by the correlation constraint.

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Appendix 1. Proofs

Proof of Proposition 1 The analysis is a two phase procedure: first, find the optimal portfolio vector on the boundary of the ellipse $\|\sigma'\boldsymbol{\pi}\| = \varepsilon$, and then find the optimal ellipse parameter ε. Let us restrict the optimization domain of the minimization problem to the ellipse boundary, $\|\sigma'\boldsymbol{\pi}\| = \varepsilon$, and find the optimal portfolio vector on this set. Thus, in the first step the objective function is:

$$\mathrm{CaR}(\boldsymbol{\pi}, \alpha, T) = -\mathbf{b}'\boldsymbol{\pi} T + \frac{1}{2}\varepsilon^2 T - z_\alpha \varepsilon\sqrt{T}. \tag{A1}$$

In order to get the minimal CaR value on the boundary of the ellipsoid, we need to maximize the linear term $\mathbf{b}'\boldsymbol{\pi}$ over this set. From the Cauchy–Schwarz inequality:

$$\begin{aligned}\mathbf{b}'\boldsymbol{\pi} = (\sigma'(\sigma\sigma')^{-1}\mathbf{b})'(\sigma'\boldsymbol{\pi}) &\le \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\| \left\|\sigma'\boldsymbol{\pi}\right\| \\ &= \left\|\sigma'(\sigma\sigma')^{-1}b\right\| \varepsilon.\end{aligned}$$

The equality is attained when $\sigma'\boldsymbol{\pi}_\varepsilon = \frac{\varepsilon}{\|\sigma'(\sigma\sigma')^{-1}b\|}\sigma'(\sigma\sigma')^{-1}\mathbf{b}$. Substituting this choice of portfolio into (A1) leads to:

$$\begin{aligned}\mathrm{CaR}(\boldsymbol{\pi}^*_\varepsilon, \alpha, T) &= \frac{\varepsilon^2 T}{2} - \varepsilon T\left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|\right) \\ &= \frac{\varepsilon T}{2}\left[\varepsilon - 2\left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|\right)\right].\end{aligned} \tag{A2}$$

It is minimized in ε by

$$\varepsilon^* = \left(\frac{z_\alpha}{\sqrt{T}} + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|\right)^+,$$

and thus we can derive equations (1) and (2) by substituting ε^* into (A2).

Proof of Theorem 1 The objective function

$$\left(-\mathbf{b}'\boldsymbol{\pi} T + \frac{1}{2}\left\|\sigma'\boldsymbol{\pi}\right\|^2 T - z_\alpha \left\|\sigma'\boldsymbol{\pi}\right\|\sqrt{T}\right)$$

is convex because of the assumption $z_\alpha \le 0$. The constraint set

$$\delta\left\|\sigma'\boldsymbol{\eta}\right\| \left\|\sigma'\boldsymbol{\pi}\right\| + \boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\eta} \le 0,$$

is also a convex set in $\mathbb{R}^m$, being the level set of a convex function. Now, we can form the Lagrangian of the convex problem:

$$\begin{aligned}L(\boldsymbol{\pi}, \lambda) = &-\mathbf{b}'\boldsymbol{\pi} T + \frac{1}{2}\left\|\sigma'\boldsymbol{\pi}\right\|^2 T - z_\alpha\left\|\sigma'\boldsymbol{\pi}\right\|\sqrt{T} \\ &+ \lambda(\delta\left\|\sigma'\boldsymbol{\eta}\right\|\left\|\sigma'\boldsymbol{\pi}\right\| + \boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\eta}).\end{aligned}$$

The Lagrangian is minimized in a two phase procedure: first find the optimal $\boldsymbol{\pi}$ on the boundary of an ellipse; then find the optimal ellipse parameter.

$$\begin{aligned}\min_{\boldsymbol{\pi}} L(\boldsymbol{\pi}, \lambda) &= \min_{\varepsilon \ge 0} \min_{\|\sigma'\boldsymbol{\pi}\| = \varepsilon} L(\boldsymbol{\pi}, \lambda) \\ &= \min_{\varepsilon \ge 0} \min_{\|\sigma'\boldsymbol{\pi}\| = \varepsilon}\Bigg[-\mathbf{b}'\boldsymbol{\pi} T + \frac{1}{2}\varepsilon^2 T - z_\alpha\varepsilon\sqrt{T} \\ &\qquad + \lambda(\delta\left\|\sigma'\boldsymbol{\eta}\right\|\varepsilon + \boldsymbol{\pi}'\sigma\sigma'\boldsymbol{\eta})\Bigg].\end{aligned} \tag{A3}$$

In the first step one has to solve:

$$\text{maximize} \quad \left[\mathbf{b}'\boldsymbol{\pi} T - \lambda\boldsymbol{\eta}'\sigma\sigma'\boldsymbol{\pi}\right] \text{ subject to } \quad \left\|\sigma'\boldsymbol{\pi}\right\| = \varepsilon.$$

By Cauchy–Schwarz inequality, we have:

$$\begin{aligned}\mathbf{b}'\boldsymbol{\pi} T - \lambda\boldsymbol{\eta}'\sigma\sigma'\boldsymbol{\pi} &= (\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda\sigma'\boldsymbol{\eta})'(\sigma'\boldsymbol{\pi}) \\ &\le \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda\sigma'\boldsymbol{\eta}\right\|\varepsilon.\end{aligned}$$

The equality occurs at

$$\sigma'\boldsymbol{\pi}_\varepsilon = \frac{\varepsilon\sigma'}{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda\sigma'\boldsymbol{\eta}\right\|}((\sigma\sigma')^{-1}\mathbf{b}T - \lambda\boldsymbol{\eta}). \tag{A4}$$

By substituting (A4) back into the (A3), the second minimization problem reduces to:

$$\min_{\varepsilon \ge 0}\left[\frac{1}{2}\varepsilon^2 T - z_\alpha\varepsilon\sqrt{T} + \lambda\delta\varepsilon\left\|\sigma'\boldsymbol{\eta}\right\| - \varepsilon\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda\sigma'\boldsymbol{\eta}\right\|\right],$$

which is solved by

$$\varepsilon^*(\lambda) = \frac{1}{T}\left(z_\alpha\sqrt{T} - \lambda\delta\left\|\sigma'\boldsymbol{\eta}\right\| + \left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda\sigma'\boldsymbol{\eta}\right\|\right)^+. \quad \text{(A5)}$$

Then the optimal portfolio satisfies

$$\sigma'\boldsymbol{\pi}^* = \frac{\varepsilon^*(\lambda^*)\sigma'}{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}T - \lambda^*\sigma'\boldsymbol{\eta}\right\|}\left((\sigma\sigma')^{-1}\mathbf{b}T - \lambda^*\boldsymbol{\eta}\right). \quad \text{(A6)}$$

Since the problem is convex, the duality gap between primal and dual problems is zero. Therefore, from the *Slater*'s condition (check Boyd and Vandenberghe (2009)), $(\lambda^*, \boldsymbol{\pi}^*)$ has to satisfy

$$\lambda^*(\delta\left\|\sigma'\boldsymbol{\eta}\right\|\left\|\sigma'\boldsymbol{\pi}^*\right\| + \boldsymbol{\pi}^{*'}\sigma\sigma'\boldsymbol{\eta}) = 0. \quad \text{(A7)}$$

Moreover, the inequality constraint of the minimization problem must hold at $(\lambda^*, \boldsymbol{\pi}^*)$:

$$\delta\left\|\sigma'\boldsymbol{\eta}\right\|\left\|\sigma'\boldsymbol{\pi}^*\right\| + \boldsymbol{\pi}^{*'}\sigma\sigma'\boldsymbol{\eta} \le 0. \quad \text{(A8)}$$

One can readily check that by substituting (A6) into (A7) and (A8), and using the fact that $\mathbf{b}'\boldsymbol{\eta} > 0$, the case $\lambda^* = 0$ and $\boldsymbol{\pi}^* \ne \mathbf{0}$ never happens. Unless $\boldsymbol{\pi}^* = \mathbf{0}$, the correlation inequality constraint always binds at $\boldsymbol{\pi}^*$, equivalently $\lambda^* > 0$. Thus, to get the non-trivial optimal primal and dual variables, we continue by taking $\lambda^* > 0$, and simplifying the equality induced by cancelling λ^* from both sides of (A7):

$$\lambda^{*2}\left\|\sigma'\boldsymbol{\eta}\right\|^4 - 2\lambda^*\left\|\sigma'\boldsymbol{\eta}\right\|^2\mathbf{b}'\boldsymbol{\eta}T + \frac{T^2}{1-\delta^2} \times\left((\mathbf{b}'\boldsymbol{\eta})^2 - \delta^2\left\|\sigma'\boldsymbol{\eta}\right\|^2\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\right) = 0,$$

which has the positive solution

$$\lambda^* = \frac{1}{\left\|\sigma'\boldsymbol{\eta}\right\|^2}\left(\mathbf{b}'\boldsymbol{\eta}T + \frac{T\delta}{\sqrt{1-\delta^2}}\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\left\|\sigma'\boldsymbol{\eta}\right\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2}\right).$$

Substituting λ^* into (A5) and (A6) completes the proof.

Proof of Proposition 2 The proof follows from theorem 1. Given the volatility matrix and its inverse in (9), and the prescribed portfolio vector for the benchmark process, one can readily find:

$$\left\|\sigma'\boldsymbol{\eta}\right\| = \left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|,$$
$$\mathbf{b}'\boldsymbol{\eta} = \left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|^2,$$
$$\sqrt{\left\|\sigma'(\sigma\sigma')^{-1}\mathbf{b}\right\|^2\left\|\sigma'\boldsymbol{\eta}\right\|^2 - (\mathbf{b}'\boldsymbol{\eta})^2} = \left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|\left\|\sigma_{22}^{-1}\mathbf{b}_2 - \sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1}\mathbf{b}_1\right\|.$$

By plugging these equations into the formulas for $\sigma'\boldsymbol{\pi}^*$ and λ^* of theorem 1, and using the invertibility of volatility matrix, we have equations (10) and (11).

Proof of Proposition 3 Let us denote by:

$$\theta_1 = \left\|\sigma_{11}^{-1}\mathbf{b}_1\right\|,$$

and

$$\theta_2 = \left\|\sigma_{22}^{-1}\mathbf{b}_2 - \sigma_{22}^{-1}\sigma_{21}\sigma_{11}^{-1}\mathbf{b}_1\right\|.$$

Direct computations lead to:

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{\theta_2^2+\theta_1^2}\right)^+\right]^2.$$

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}_c{}^*}(T)) = T\left[\left(\frac{z_\alpha}{\sqrt{T}} + \sqrt{1-\delta^2}\theta_2 - \delta\theta_1\right)^+\right]^2.$$

In the light of the following inequality for $\delta \in [0, 1]$:

$$\sqrt{\theta_2^2+\theta_1^2} \ge \sqrt{1-\delta^2}\theta_2 - \delta\theta_1,$$

it follows that

$$\mathrm{Var}(\log X^{\boldsymbol{\pi}^*}(T)) \ge \mathrm{Var}(\log X^{\boldsymbol{\pi}_c^*}(T)).$$