Reputation, Innovation, and Externalities in Venture Capital

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Abstract

I introduce a dynamic model of random search where ex ante heterogeneous venture capitalists (investors) with unknown abilities match with a variety of startups (projects). There is incomplete yet symmetric information about investors’ types, whereas the projects’ types are publicly observable to all investors. In the unique stationary equilibrium, the matching sets, value functions and steady state distributions are endogenously determined. Interpreting the market posterior belief about the venture capitalists’ ability as their reputation, I study the outcomes of the economy when the success or failure of the projects create feedback effects: innovation spillovers and reputational externalities. When there are positive spillovers from successful early stage projects to late stage business opportunities, I show increased levels of search frictions could save the market from breakdown caused by the neglect of spillover effect. When the reputational externality is at play, namely when the deal flow of each investor is inversely impacted by the distribution of other investors’ reputation, I show the proportion of the high ability inactive investors is inefficiently high, and the projects suffer from early termination.

JEL classification: C78; D83; G24; M13; O31

Keywords: Reputation; Learning; Search and Matching; Venture Capital

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1 Introduction

There have been a number of successful and failed public attempts to promote entrepreneurship, venture capital and innovative finance over the past half-century (Lerner (2012)). The fundamental rationales for such interventions have been to correct the market failures originating from the existence of information gaps between investors and businesses, and to internalize the positive externalities in the innovative sector. In this paper, I focus on the latter reason as well as a novel one, the impact of each VC’s reputation on the deal flow of the other investors. Both of these move the decentralized outcome away from the social optimum. I argue that any policy aimed to moderate the extent of such market failures in the venture capital industry should take into account two inevitable strains: the search frictions due to the absence of a centralized investor/investee market; and the lack of complete information about the ability of the investors, prompting the market participants to form rational beliefs and thus rely on the investors’ reputation. I show that, as a result, the investors decisions are endogenously tilted along the two margins of search frictions and reputation.

On the first margin: when making investment decisions, venture capitalists rationally take into account the opportunity cost of forgoing the investment in the late stage businesses in favor of the early stage ones. Correspondingly, holding everything else constant, higher search frictions decrease this opportunity cost, and hence raise the likelihood of investment in early stage businesses. This effect becomes more prominent when the spillovers from early stage businesses to late stage counterparts are taken into account. Specifically, I show that there are regions where higher search frictions could save the market from a total breakdown created by the individually rational neglect of VCs to invest in early stage companies in the hope of receiving a better proposal from a late stage project.

On the second margin: financing expenses for running the startups – that translate to costly learning of investors’ types – create a group of investors who have low reputation in spite of their high ability (henceforth referred to as dormant investors). I show when higher reputation engenders higher deal flow, the size of the dormant group is sub-optimal. This is because the more reputable investors impose a negative externality on the deal flow of the less reputable ones, which in turn pushes down the value of reputation building for the latter group, thereby increasing the size of the dormant investors. This pattern is also associated with the early termination of projects that kills the startups earlier than a constrained-efficient scheme suggests.
Next, I explain the relevant forces behind each of these two margins, and continue the introduction by shedding more light on the two points raised above.

**Search frictions.** The market in which venture capitalists operate as the investors and entrepreneurs as the investees is certainly far from a centralized market in which prices equilibrate the demand and supply for capital. For example, there are geographical barriers hindering the connection of remote startups to the centers of VC finance. Even within the financial centers there are informational barriers vis-à-vis availability of the capital for startups and investment opportunities for investors. The VCs create and participate in syndication networks that facilitate the exchange of information about investment opportunities (Sørensen (2007)). However, there still remains a significant amount of unexploited partnerships between startups and VCs that would have otherwise been formed in an informationally and spatially centralized market. These observations motivate us to select the framework of dynamic matching and random search as the basis of the economy that will be studied in this paper. Specifically, the agents of our *economy* are VCs and startups who randomly meet each other and form partnerships. The speed of such random meetings parametrizes the extent of search frictions in the economy.

**Investors’ ability and incomplete information.** In all industries and especially in the venture capital industry, investors add value to the projects through several channels. The first and foremost one is funding the project, but the focus of this paper is on the other values they provide. A fair body of previous research has pointed out these value-added services. For example, in the surveys done by Hellmann and Puri (2002) and Gompers et al. (2020), VCs responded that they provide a range of post-investment services to their portfolio companies. Two notable studies of Sørensen (2007) and Bernstein et al. (2016) have gone even further by teasing out and identifying the positive treatment effect of the VCs’ involvement in their portfolio companies, from the concerns regarding the sorting and selection effects. All in all, the evidence suggests the investors’ ability (or lack thereof) has an impact on the success likelihood of their portfolio companies. However, one can not expect that this ability is held by all investors, and naturally some may fail to possess such

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A testament to that is the substantial concentration of startups and VC funds in few states. According to the data released by NVCA, companies headquartered in three states of California, Massachusetts and New York collectively account for 73% and 79% of the total venture capital spending in 2019 and 2020, respectively. Also, the funds based in these three states represent 92% and 86% of the total venture capital raised in the US, respectively in 2019 and 2020.

The list includes strategic and operational guidance, connecting investors and customers, hiring employees, as well as supervising startup professionalization measures such as HR policies and the adoption of stock option plans. Further studies such as Gorman and Sahlman (1989) and Lerner (1995) highlight the VCs involvement and their oversight on the boards of private firms in their portfolios.
qualities. Therefore, I assume that investors are endowed with types, indicating their ability, whereby a high type indicates a high-skilled investor and a low type refers to an investor who lacking the aforementioned expertise. At this stage, a crucial decision needs to be made. One needs to choose one of the two relevant information structures: (i) a *learning* model in which neither the VC nor the other market participants know the VC’s type; or a (ii) *signalling* model in which the VC is aware of his type and the rest are not. Backed by the empirical validation of Gompers and Lerner (1999), I choose the former model, and assume the presumptive VC and the rest of the market have *incomplete yet symmetric information* about each VC’s type.

The only publicly verifiable signal, resolving the uncertainty about each VC’s type, is the occurrence of the successful outcome in their respective companies. Therefore, whenever a VC pairs up with a startup, a learning opportunity is created for the entire market as well as that particular VC about its type. However, learning is costly, because the associated VC must finance a startup for the learning process to take place. Therefore, at every time a partnership is formed the corresponding VC is going to solve a *stopping time* problem, by which it weighs the value of the match (as a function of its current reputation and the type of the partnering startup) against the reservation value – the value of holding current reputation while not being matched to a startup, that is called the reputation value function throughout the paper. In equilibrium these two value functions are intertwined, and jointly determine the matching sets.

I show that within the space of increasing value functions (as a function of reputation), there is a unique equilibrium. The equilibrium matching sets encode the investment decisions of VCs. Namely, they specify what types of businesses (early versus late stage, or radical versus incremental ideas) an investor with a certain level of reputation chooses to invest. As a result, I provide a framework to endogenize the so-called measure of *tolerance for failure*, that determines the extent to which VCs exhibit patience on the project they fund (Manso (2011) and Tian and Wang (2014)). Specifically, in the equilibrium, investors with higher reputation exhibit higher tolerance, as the distance of their current reputation to the endogenous separation point of the match is larger. Furthermore, when it comes to cross-company comparison, they show higher patience toward the late stage companies. Also, the equilibrium outcome predicts that, as the cost of financing the startups falls – due to the technological developments, the investors increase the *variety* of the projects they finance. Namely, they start to channel their capital to the early stage projects as well as the late stage ones. This prediction confirms the prevalence of the investment approach, “*spray and
Leveraging the baseline model, I study the outcome of the economy when there are spillovers from successful early stage projects to the late stage investment opportunities. One can alternatively interpret this in the context of knowledge spillovers from radical innovations to incremental ones. At any rate, there are empirical evidences suggesting that small innovative firms are particularly weak in protecting their intellectual property and/or extracting all of their created social rents. Therefore, one should naturally expect not all of the future gains created by investing in early stage ventures are internalized in the decisions of their respective investors, and hence the decentralized outcome of the economy inevitably exhibits under-investment in this group of companies. By solving the social optimum in the planner’s problem, I obtain the magnitude and the direction of the decentralizing transfers which satisfy budget neutrality and restore the market efficiency. The optimal redistribution policy features a tax on the reputation value function – the investors’ valuation as a function of their reputation while not investing in projects – and a subsidy to the early stage investors.

Next, building on the baseline model, I study the outcome of the economy when there is a reputational externality at play. In particular, I study the direction along which the social optimum of the economy departs from the equilibrium outcome. I show when the matching function (between VCs and startups) accounts for higher deal flow due to higher reputation – via the means of a reputation weight function – the more reputable investors slow down the deal flow of the less reputable ones, thereby making the latter group less patient by lowering their value of reputation building. This amounts to under-learning and early termination of projects by the novice investors. Specifically, it turns out the equilibrium threshold to terminate the funding from the businesses would be tighter than what is socially optimal. Through a comparative static exercise on the choice of the reputation weight function, I further show as the effect of the reputation on the deal flow wanes, the threshold to terminate the funding and end the partnerships tightens, while the opportunity cost of investing in early stage projects falls.

The results set forth above on the relation between reputation and deal flow are related to a substantial body of previous research that studies the individual benefits associated with higher reputation. The findings include the theory of grandstanding, and lower pay-for-performance for smaller and younger VC firms toward the goal of establishing a reputation and enjoying a higher deal flow (Gompers (1996) and Gompers and Lerner (1999)); Or how VCs with higher reputation acquire startup equity at a discount (Hsu (2004)). Relatedly, the public policy report about the New Zealand government’s efforts to stimulate the venture capital industry by Lerner et al. (2005) highlights many of these issues.
by dissecting investment-level data Nanda et al. (2020) finds that initial success confers preferential access to deal flow and perpetuates the early superior performances made by successful VCs. However, in contrast to what I study in this paper, none of these previous studies investigates the social return and the aggregate outcome when the deal flow of a single investor depends on the average reputation weight of the remaining body of the investors.

**Other related literature.** This paper is also related to a group of other works, Silveira and Wright (2016), Ewens et al. (2019) and Sannino (2019), that study the search and matching between VCs and entrepreneurs in an environment where VCs’ types are perfect knowledge thus there is no room for reputation building and learning. On the macroeconomic impact of VC sector, Opp (2019) develops an endogenous growth model in which VCs’ intermediation in conjunction with entrepreneurs’ ideas and labor contribute to the aggregate growth. In another work, Akcigit et al. (2019) develops a static equilibrium model with perfect information on agents’ types that captures and estimates the positive assortative matching between entrepreneurs and VCs.

**Organization of the paper.** In section 2, I present the baseline model of the economy with VCs and startups as the agents and meetings subject to search frictions. The equilibrium value functions and matching sets are determined and the investors trade-offs vis-à-vis projects are explained. In section 3, I express the economy’s social surplus and verify the constrained-efficiency. The learning outcome of the economy, namely the extent to which the decentralized outcome can uncover the venture capitalists’ types is studied in section 4. In section 5, the inflow of the late stage projects in the economy are endogenized by letting them to be proportional to the mass of successful early stage projects. In addition, the first case of market failure, in which investors fail to internalize the spillovers from early to late stage projects, is shown in this section. In section 6, the matching function exhibits the reputational externality, accounting for the fact that higher reputation increases the deal flow. In light of this externality, I establish the theoretical grounds behind the second case of market failure, and show the direction along which the equilibrium outcome departs from the social optimum. The paper concludes in section 7, and all the proofs and verifications that are not stated in the main body are relegated to the appendix.
2 Equilibrium in the baseline economy

In this section, I describe the constituents of an economy populated by a unit mass of long-lived venture capitalists (investors) and a continuum of startup projects (investees).

Investors (supply side). The agents in this side of the market are the long-lived investors, who care about their reputation, which is the market posterior belief about their type \( \theta \in \{L, H\} \). Throughout the paper I take venture capitalists as the leading example for the investor side. Given the market filtration \( I = \{I_t\} \), \( \pi_t = P(\theta = H|I_t) \) refers to the time \( t \) reputation of a generic VC. The \( \sigma \)-field \( I_t \) aggregates all information that market participants hold at time \( t \in \mathbb{R}_+ \). The share of high-type VCs is equal to \( p \), exogenously set and publicly known.

Investees (demand side). The entities on the demand side of the market are simply treated as investment opportunities that are chosen by the investors. Specifically, they have no bargaining power against investors. The leading case for investees throughout the paper are the startups. Each startup is endowed with a type \( q \in \{a, b\} \), which is publicly observable. The (unnormalized) mass of type-\( q \) projects is \( \varphi_q \) for \( q \in \{a, b\} \).

Matching and partnerships. Pairwise meetings between agents in two sides of this market take place. The meetings are subject to search frictions with the meeting rate \( \kappa > 0 \), and the matching technology is quadratic à la Chade et al. (2017) (and the references therein), that is the probability with which a generic VC meets a type-\( q \) startup over the period \( dt \) is \( \kappa \varphi_q dt \). Furthermore, the matches are one to one, that is both parties have capacity constraint over the number of partners they can match with.

Output and reputation. Given the partnership between a type-\( \theta \) VC and a type-\( q \) startup, the success arrives at the rate \( \lambda_q(\theta) \), where \( \lambda_q(H) = \bar{\lambda}_q \) and \( \lambda_q(L) = \underline{\lambda}_q \), with normalized payoff of one. The VC has to cover the flow cost of project \( c > 0 \) that is common across all matches. In return, she receives the right to terminate the funding at her will, so conceptually a stopping time problem is solved by each VC ex post to every partnership formation. I

This assumptions makes the analysis substantially simpler, yet it downplays the strategic role of startups in the equilibrium outcomes. However, given the paper's focus on the VC side and their reputational concerns such an abstraction seems plausible. Also from the empirical standpoint there are evidences suggesting that venture-backed firms can continue their projects without their original entrepreneurs; see Ueda (2004) and the references therein such as Gorman and Sahlman (1989) and Hellmann and Puri (2002).
assume there is a mechanism in the market that tracks the output of each partnership and records the Bayes-updated posterior of every VC during its match. This information is reflected in the market filtration $\mathcal{I}$. The posterior dynamics for the reputation process thus follows

$$d\pi_t = -\pi_t(1 - \pi_t)\Delta_q dt,$$

prior to the success, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. For the purpose of simplicity, I assume the breakthroughs are conclusive in the sense that $\Delta_q = 0$, that is the success never happens to a low type VC, therefore upon the success event $\pi_t$ immediately jumps up to one. Further, without loss of generality it is assumed $\lambda_b := \bar{\lambda}_b > \lambda_a := \bar{\lambda}_a$. Also, I assume $p > c/\lambda_b$ throughout, because otherwise there are cases in which even the high-type projects are not worth the investment.

Figure 1 summarizes the dynamic timeline of the investment path for a generic venture capitalist, who starts the cycle with reputation $\pi$, and after some exponential random time meets a startup randomly drawn from the population of unmatched entrepreneurs. A decision to accept or reject the contacted startup is made by the VC; Upon rejection the VC returns to the initial node, and conditioned on acceptance an investment problem with the flow cost of $c$ is solved by the VC. Finally, a success or a failure at the terminal node guides the entire market participants to rationally update their beliefs about the ongoing VC, and the associated VC returns back to the pool of available investors.

2.1 Value functions and matching sets

The rate of time preference for investors in this economy is $r > 0$. Let $w(\pi)$ be the value of holding reputation $\pi$, when the VC is unmatched. This function shall be treated as the VC’s outside option and is weighed against the matching value function upon the meetings. The matching value function when a reputation-$\pi$ VC pairs up with a type-$q$ startup is $v(\pi, q)$, that is the expected value of discounted future payoffs generated by this partnership. Therefore, a match is profitable if $v(\pi, q) > w(\pi)$, in that case I say $(q, \pi) \in \mathcal{M} \subseteq \{a, b\} \times [0, 1]$, where $\mathcal{M}$ is called the matching set. Also, understood from the context, $\mathcal{M}(\pi)$ (resp. $\mathcal{M}_q$) refers to the $\pi$ (resp. $q$) section of this set. In addition, often in the paper I use the

In the supplementary appendix B, I relax this assumption and study the general case, where the success is not necessarily conclusive and there is a continuum of projects with the type space $[a, b]$ distributed according to an arbitrary CDF function $\phi$. On a further note, the notion of conclusive breakthroughs is studied by Keller et al. (2005) in the context of strategic experimentation, and in a follow-up paper Keller and Rady (2015) highlight the technical contrasts with the case of inconclusive breakthroughs.

That is for example $\mathcal{M}(\pi) = \{q : (q, \pi) \in \mathcal{M}\}$ and $\mathcal{M}_q = \{\pi : (q, \pi) \in \mathcal{M}\}$.
indicator function $\chi_q(\pi)$ to denote whether a reputation-$\pi$ VC matches with a $q$-startup, that is whether $(q, \pi) \in \mathcal{M}$ or not. Recall that $\varphi$ denotes the stationary mass of available startups in the economy (that are so far treated exogenously as the primitives of the model). Below, I invoke a standard dynamic programming analysis for $w(\pi)$:

$$w(\pi) \approx \kappa \sum_{q \in \mathcal{M}(\pi)} (w(\pi) + [v(\pi, q) - w(\pi)]) \varphi_q dt + \kappa \sum_{q \in \{a, b\} \setminus \mathcal{M}(\pi)} w(\pi) \varphi_q dt$$

$$+ (1 - \kappa \varphi(\{a, b\}) dt) (1 - rd(\pi))$$  \hspace{1cm} (2.2)$$

The first term in the rhs is the expected value of payoffs generated from all acceptable matches, taking into account that the next project with type $q$ arrives at the rate of $\kappa \varphi_q$. The second term is the expected payoff over all denied partnerships, and the third term simply refers to the discounted payoff conditioned on receiving no investment proposal over the period $dt$. Accounting for these three sources, the following Bellman equation for the reputation value function $w$ is resulted:

$$rw(\pi) = \kappa \sum_{q \in \mathcal{M}(\pi)} [v(\pi, q) - w(\pi)] \varphi_q$$  \hspace{1cm} (2.3)$$

Next, I examine the matching value function $v(\pi, q)$. Imagine a partnership of a VC with initial reputation $\pi$ and a type-$q$ startup. Let $\sigma$ represent the random exponential time of success with unit payoff and the arrival intensity of $\lambda_q$ if $\theta = H$. Therefore, the matching value function $v(\cdot, q)$ is an endogenous outcome of a free-boundary problem with the outside option $w$. In that the VC selects an optimal stopping time $\tau$, upon which she stops funding.

Figure 1: Investment timeline for a generic VC
the project, taking into account the project’s success payoff and her reputation value \( w \):

\[
v(\pi, q) = \sup_{\tau} \left\{ \mathbb{E} \left[ e^{-r\sigma} - c \int_0^\sigma e^{-rs} ds + e^{-r\sigma} w(\pi_\sigma); \sigma \leq \tau \right] + \mathbb{E} \left[ -c \int_0^\tau e^{-rs} ds + e^{-r\tau} w(\pi_\tau); \sigma > \tau \right] \right\}
\]

(2.4)

The exit option upon the stopping time \( \tau \) is the VC’s reservation value of holding reputation \( \pi_\tau \). The corresponding HJB representation for this stopping time problem is

\[
rv(\pi, q) = \max \{ rw(\pi), -c + \lambda_q \pi (1 + w(1) - v(\pi, q)) - \lambda_q \pi (1 - \pi) v'(\pi, q) \}.
\]

(2.5)

The above HJB is presented in the variational form, that is the first expression in the rhs is the value of stopping – refusing the match and holding on to the outside option \( w \) – and the second expression represents the Bellman equation over the continuation region \( \mathcal{M}_q \), on which \( v(\pi, q) > w(\pi) \). The first term in the Bellman expression is the flow cost of the project borne by the VC, the second term is the created surplus upon the success event that is the unit payout added to the value of holding the maximum reputation \( \pi = 1 \) minus the current value of the match, and the last term captures the marginal reputation loss due to the lack of success. Induced by the above stopping time problem, the matching set \( \mathcal{M} \) can thus be interpreted as the continuation set for the free-boundary problem (2.5), namely

\[
\mathcal{M} = \{(q, \pi) \in \{a, b\} \times [0, 1] : v(\pi, q) > w(\pi) \},
\]

(2.6)

and on the stopping region \( \mathcal{M}^c \), the matching value function equals the VC’s reputation function, i.e \( v(\pi, q) = w(\pi) \).

### 2.2 Equilibrium construction

The goal of this section is to progressively suggest the necessary conditions pinning down the equilibrium outcome and finally express the properties of the matching sets in equilibrium.

**Definition 1** (Stationary equilibrium). Given the mass function \( \varphi \) for the unmatched startups, the tuple \( \langle w, v, \mathcal{M} \rangle \) constitutes a stationary equilibrium, if (i) given \( v \) and \( \mathcal{M} \), the reputation value function \( w \) satisfies (2.3); (ii) Given \( w \), the matching value function \( v \) and the matching set \( \mathcal{M} \) together solve the free boundary system (2.5) and (2.6).

The two-way feedback between the reputation function \( w \) and the matching variables
⟨v, M⟩ are portrayed in figure 2. The link connecting w to the ⟨v, M⟩ block is upheld by the stopping time problem (2.4), and its recursive representation (2.5). The reverse link from the matching variables block to w is supported by the Bellman equation for the reputation function (2.3). Then, the stationary equilibrium is formally the fixed-point to the endogenous loops of figure 2.

![Equilibrium feedbacks](image)

Figure 2: Equilibrium feedbacks

Next lemma uses (2.3) to express the reputation value function in terms of v and M, and thereby provides a partial characterization of matching sets only in terms of the matching value functions. Its proof is offered in the supplementary material C.

**Lemma 2.** A VC with reputation π accepts both types of companies, namely π ∈ Ma ∩ Mb iff

\[
v(\pi, a) \left(1 - \frac{1}{1 + r^{-1}K\phi_a}\right) < v(\pi, b) < v(\pi, a) \left(1 + \frac{1}{r^{-1}K\phi_b}\right).
\]  

(2.7)

In addition, π ∈ Mb ∩ Mc_a iff the upper-bound is achieved, π ∈ Ma ∩ Mc_b iff the lower bound is achieved, and π ∈ Mc_a ∩ Mc_b iff the upper and lower bounds coincide, which is only the case where all value functions are zero.

Intuitively, this lemma asserts that the ratio \(v(\pi, b)/v(\pi, a)\) always lies in a bounded interval for \(\pi \in M_a \cup M_b\). At its maximum where it reaches the upper bound, the VCs do not invest in a-projects and alternatively, when it hits the lower bound, the investors only choose the a-startups. This analysis renders much of the results in the next proposition on the equilibrium shape of the matching sets.

Throughout the paper I seek to construct equilibria with increasing value functions in π. Specifically, in the baseline model and its proceeding extensions the focus is given to increasing functions \(v(\cdot, q)\) and \(w(\cdot)\) in π. Toward this construction, let us rewrite equation (2.3) as

\[
w(\pi) = \frac{r^{-1}K[v(\pi, a)\phi_a\chi_a(\pi) + v(\pi, b)\phi_b\chi_b(\pi)]}{1 + r^{-1}K[\phi_a\chi_a(\pi) + \phi_b\chi_b(\pi)]}.
\]  

(2.8)

First, note that the proof of lemma 2 (presented in supplementary section C) will fall out of a case by case guess and verify over the relative orderings of \(v(\pi, a), v(\pi, b)\) and \(w(\pi)\). Second,

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I use the word increasing to refer to a non-decreasing function.
this representation of \( w(\pi) \) and lemma 2 allow us to express the equilibrium reputation value function \( w \) as the output of a maximization problem over the space of all Borel measurable indicator functions \( \chi_q(\pi) \) (similar idea to lemma 1 of Shimer and Smith (2000)):

\[
w(\pi) = \max_x \left\{ \frac{r^{-1} [v(\pi,a)\varphi_a(\pi) + v(\pi,b)\varphi_b(\pi) + \varepsilon]}{1 + r^{-1} [\varphi_a(\pi) + \varphi_b(\pi)]} \right\}
\]

(2.9)

An important consequence of the above representation is that if \( v(\cdot, a) \) and \( v(\cdot, b) \) are increasing in \( \pi \), then it would be case that \( w(\cdot) \) is increasing in \( \pi \) as well. The reverse direction is the result of the following lemma. This lemma assures us that in any equilibrium an increasing pair of matching value functions lead to an increasing reputation value function and vice-versa.

**Lemma 3.** *The matching value functions \( \{v(\cdot, q) : q \in \{a, b\}\} \) are increasing in \( \pi \) if and only if \( w(\cdot) \) is increasing in \( \pi \).*

Continuing the path toward equilibrium construction, I would now analyze the Bellman equation for the matching value functions. In the sequel, I repeatedly use the general solution form for the Bellman equation (2.5) on the continuation region \( \mathcal{M}_q \), in that \( \varpi(q) \) is the constant dependent on the appropriate boundary conditions:

\[
v(\pi, q) = -\frac{c}{r} + \frac{\lambda_q}{r + \lambda_q} \left( 1 + w(1) + \frac{c}{r} \right) \pi + \varepsilon(q) (1 - \pi)^{1+r/\lambda_q} \pi^{-r/\lambda_q}
\]

(2.10)

To further examine the essence of the stopping time (2.5), I point out to two necessary conditions that the optimal matching value function and the continuation region must satisfy. The dynamics of the reputation process can be compactly represented by \( d\pi_t = (1 - \pi_t - \mu_t) dt - \lambda_q \pi_t - dt \), in that \( t \) is the success indicator process, that is \( t := 1_{\{t \geq \sigma\}} \). The infinitesimal generator associated to this stochastic process is \( \mathcal{L}_q : C^1[0,1] \to C^1[0,1] \), where for a generic \( u \in C^1[0,1] \):

\[
[\mathcal{L}_q u](\pi) = \lambda_q \pi (1 + w(1) - u(\pi)) - \lambda_q \pi (1 - \pi) u'(\pi)
\]

(2.11)

For every candidate equilibrium tuple \( \langle w, v, \mathcal{M} \rangle \), the following two conditions must hold for all \( \pi \in [0,1] \) and \( q \in \{a, b\} \):

(i) **Majorant property:** \( v(\pi, q) \geq w(\pi) \).

These two conditions are standard in the literature of optimal stopping time and can be found in chapter 2 of Peskir and Shiryaev (2006).

Space of continuously differentiable functions on \([0,1]\).
(ii) **Superhamonic property:** \( [\mathcal{L}v](\pi,q) - rv(\pi,q) - c \leq 0 \).

The first condition simply means that in every partnership the VC has the option to terminate the funding, thus enjoying her reputation value \( w \) by severing the match. The second condition means *on expectation* a generic VC loses if she decides to invest on the stopping region. The following proposition establishes a set of descriptive properties of equilibrium when the value functions are increasing and belong to \( C^1[0,1] \). In doing so, it is important to recall that because of continuity of value functions the sections of matching sets, \( \mathcal{M}_a \) and \( \mathcal{M}_b \), are open subsets of \([0,1]\). So, to characterize them, it is sufficient to identify their boundary points. For this I employ lemma 2 and the above two optimality conditions in conjunction with \( \lambda_b > \max\{\lambda_a, c\} \) to identify these boundary points, thereby the equilibrium shape of the matching set. As it turns out there appear two distinct equilibrium regimes, *low* and *high cost*, that respectively correspond to \( \lambda_a - c > \frac{\kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b} \) and \( \lambda_a - c \leq \frac{\kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b} \).

**Proposition 4** (Equilibrium shape of the matching sets). *In every stationary equilibrium with increasing value functions belonging to \( C^1[0,1] \), the following properties hold:*

(i) *(Status of \( \pi = 1 \)): in both regimes \( 1 \in \mathcal{M}_b \), and \( 1 \in \mathcal{M}_a \) only in the low cost regime.*

(ii) *In both regimes the matching set \( \mathcal{M}_b \) is a connected subset of \([0,1]\).*

(iii) *In the high cost regime \( \mathcal{M}_a = \emptyset \) and in the low cost regime \( \mathcal{M}_a \) is a connected subset of \( \mathcal{M}_b \).*

![Figure 3: Equilibrium matching sets](image)

Figure 3 illustrates the equilibrium matching sets in both cost regimes. There are a few points related to this result that should be raised. First, it is the comparison between the
expected flow payoff of investing in \( a \)-projects and the opportunity cost of forgoing the wait for the next \( b \)-projects that determines the cost regime:

\[
\text{low cost regime } \Leftrightarrow \lambda_a - c > \frac{\kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b} \tag{2.12}
\]

opportunity cost of forgoing the wait for a \( b \)-project

For instance, as the share of available \( b \)-projects (\( \varphi_b \)) increases, the opportunity cost of investing in \( a \)-projects increases and consequently VCs become more reluctant to invest in \( a \)-companies. Second, one can verify that lowering \( c \) increases the expected flow payoff of investment in \( a \)-startups more than it does the opportunity cost component, thereby enhancing the variety of financed projects. Therefore, the equilibrium response observed in the matching sets confirms the prevalence of the investment approach “spray and pray” that arises due to the cost-reducing technological shocks, mentioned in Ewens et al. (2018).

Third, this model suggests a method to endogenize the tolerance for failure (see Tian and Wang (2014) and Manso (2011)) by relating it to the investor’s reputation. The equilibrium observation in figure 3 on connectedness of the matching sets advances the idea that VCs with higher reputation have higher tolerance for failure. In other words, the distance to the endogenous separation point \( \alpha \) is larger for a more reputable VC than a less reputable one. Furthermore, when it comes to cross-company comparison, the investors show more patience toward \( b \)-companies – that confer faster success time on average. Fourth, in light of \( M_a \subset M_B \) the model offers the testable prediction that VCs who exit the market and do not raise subsequent funds made their last few investments in the high-growth companies (i.e \( b \)-startups). Formally, in both panels of figure 3 we see that the endogenous termination point \( \alpha \) is the lower boundary point of \( M_b \) (not \( M_a \)), at which the matching value function \( v(\cdot, b) \) smoothly meets the zero function (as shown in the proof of the last proposition). Also in the proof, it is established that in equilibrium

\[
\alpha = \frac{c}{\lambda_b (1 + w(1))}, \tag{2.13}
\]

where \( w(1) \) is the value of holding maximum reputation, i.e \( \pi = 1 \), in each cost regimes. In

Specifically, in Tian and Wang (2014) VCs learn about the quality of the startup over the course of the match, whereas in my model the startup’s quality is observable and the learning is about VCs’ type. Consequently, the scheme here suggests one way to endogenize the tolerance parameter \( \phi \) in Tian and Wang (2014).

It is shown in the proof of proposition 4, the smooth pasting and value matching at \( \alpha \) is ensued in spite of the Poissonian environment and the absence of diffusion processes.
the high cost regime \( w(1) \) only depends on the \( b \)-parameters, because \( M_a = \emptyset \), whereas in the low cost regime it takes the \( a \)-related parameters into account as well. Some easy-to-verify comparative statics (for instance in the former case) are \( \frac{\partial \alpha}{\partial c} > 0, \frac{\partial \alpha}{\partial \lambda_b} < 0 \) and \( \frac{\partial \alpha}{\partial \phi_b} < 0 \).

Having known the form of the matching sets that are sustained in the equilibrium, I can now state the main theorem related to the decentralized behavior of this economy, i.e the fixed-point outcome of figure 2.

**Theorem 5** (Stationary equilibrium, existence and uniqueness). *There exists a unique stationary equilibrium in the space of continuously differentiable and increasing payoff functions in each cost regime. Further, for large values of discount rate \( r \), this equilibrium is unique in the bigger space of continuous functions \( C[0, 1] \).*

The substantial result of this theorem is that there always exists an equilibrium tuple in which the value functions are increasing and continuously differentiable in reputation. Furthermore, there is not a possibility for multiple equilibria of such kind. However, the possibility of other equilibria with non-increasing value functions can not be ruled out unless the discount rate is large enough so that a contraction theorem type method can be applied.

### 3 Social surplus

The economy as stated thus far exhibits no externality, because the investment decision made by every venture capitalist, regardless of its reputation level, has no impact on the deal flow of other investors. First, this is owed to the fact that the mass of unmatched startups are treated exogenously, and not impacted by VCs actions. Second, the matching technology exhibits no interaction effect from one group of VCs to another. Therefore, one would expect that the equilibrium proposed in the previous section is constrained-Pareto-optimal.

To fix ideas, the planner’s problem is to maximize the social welfare, taking the matching set \( \mathcal{M} \) as the control variable:

\[
S(\mathcal{M}) = \int_{[0,1]} w(\pi)G(d\pi) + \sum_{q \in \{a,b\}} \int_{\mathcal{M}_q} v(\pi,q)F(d\pi, \{q\}),
\]

The measures \( G \) and \( F \) are the steady state distributions of unmatched and matched VCs, subject to \( w, v \geq 0 \) and be continuous increasing functions. Requiring increasing value functions means that \( \mathcal{M}_a \cup \mathcal{M}_b \) must be connected and contain \( \pi = 1 \). Connectedness since \( (\mathcal{M}_a \cup \mathcal{M}_b)^c = \{ \pi : w(\pi) = 0 \} \), then an increasing \( w \) means \( \{ \pi : w(\pi) > 0 \} = \mathcal{M}_a \cup \mathcal{M}_b \) must be a connected set.
also implies that in the steady state the matched distribution $F$ only places positive mass at $\pi = 1$, and the support of the unmatched distribution $G$ is comprised of the lowest boundary point denoted by $\alpha := \inf \mathcal{M}_a \cup \mathcal{M}_b$ and the highest boundary point 1. Note that $F(\{\alpha\}, \{a, b\})$ must be zero because the reputation process spends no time at this point, as it either immediately drops below $\alpha$ and hence not belongs to $\mathcal{M}$ anymore, or has already jumped up to 1 before reaching $\alpha$. Therefore, $\alpha \notin \mathcal{M}_a \cup \mathcal{M}_b$, and since the VCs holding such a reputation will never be rematched again then $w(\alpha) = 0$. Consequently, the steady state population of VCs can be summarized by four distinct masses: $n(\alpha)$ VCs trapped at $\alpha$, in addition to three other groups with maximum reputation, $n(1)$ unmatched, $m_a(1)$ matched to $a$-companies and $m_b(1)$ matched to $b$-companies. Inflow outflow equations together with the Bayesian consistency at the steady state amount to

\[
\begin{align*}
n(\alpha) + n(1) + m_a(1) + m_b(1) & = 1 \quad (3.2a) \\
\alpha n(\alpha) + n(1) + m_a(1) + m_b(1) & = p \quad (3.2b) \\
\kappa n(1) \varphi_a \chi_a(1) & = \lambda_a m_a(1) \quad (3.2c) \\
\kappa n(1) \varphi_b \chi_b(1) & = \lambda_b m_b(1) \quad (3.2d)
\end{align*}
\]

The first equation simply says that the total mass of VCs is one. The second equation states that in the steady state the average ability of VCs must be equal to the initial average ability $p$. The third (resp. fourth) expression equates the inflow to the group of VCs investing in $a$-startups (resp. $b$-startups) to its outflow (that is the rate at which these projects experience success thus their corresponding VCs exit their position and become unmatched). These distributional results help us in the next proposition in which I analyze the constrained-efficiency of this economy. Specifically, in the next proposition I treat the choice of the matching sets as the only instrument of a benevolent planner and prove the constrained efficiency.

The central trade-off in the choice of the matching sets in the planner’s problem is between the benefits of lowering $\alpha$, thus increasing the size of active investors in the economy, and its associated cost born by the VCs as a result of longer financing periods. Lowering $\alpha$ on one hand increases the pool of active VCs and improves the learning prospects of the economy by letting VCs to experiment longer, thereby ending up owning more certain beliefs about their skills. On the other hand, learning about their know-how abilities is costly, thus limiting the scope of perfect learning and resolution of investors’ types. The welfare result in the next proposition asserts that in the absence of externalities among VCs and the presence
of exogenous flow of startups, i.e \( \phi_a \) and \( \phi_b \) not being impacted by investors’ decisions, the economy is constrained-efficient. Both of these premises are relaxed in the proceeding sections.

**Proposition 6.** The equilibrium matching sets characterized in proposition 4 are constrained-Pareto-optimal.

## 4 Imperfect learning

The welfare analysis offered above sheds some light on the close connection between the learning outcome and the social surplus. Specifically, it was shown that there exists a mass of \( n(\alpha) \) of VCs who are inactive and no longer raise funds and take on projects, some of whom indeed have the expertise and the know-how yet failed to prove it in their first few investments. Henceforth, I refer to this group as *dormant* investors. In this section, I aim to study the steady state distribution of VCs reputation, its distance to the *perfect learning benchmark*, and its connection to the social surplus of the economy. The analysis in the proof of previous proposition as well as the shape of the matching sets suggest that in the stationary equilibrium there is a non-zero mass of VCs trapped at \( \alpha_b \), the lower-boundary point of \( M_b \). The main reason behind the existence of this group is that learning is costly, therefore at some point the cost does not rationalize the expected payoff and the learning stops.

**Distance to perfect learning.** Constrained by the search frictions the maximum created surplus, that is also achieved in the equilibrium, as found in proposition 6 follows

\[
\begin{align*}
\bar{r}_S^{HC} &= \frac{p - \alpha_{HC}}{1 - \alpha_{HC}} \frac{\kappa \phi_b / \lambda_b - \kappa \phi_b / \lambda_b}{1 + \kappa \phi_b / \lambda_b} (\lambda_b - c), \\
\bar{r}_S^{LC} &= \frac{p - \alpha_{LC}}{1 - \alpha_{LC}} \frac{1}{1 + \kappa \phi_a / \lambda_a + \kappa \phi_b / \lambda_b} \left( \frac{\kappa \phi_a}{\lambda_a} (\lambda_a - c) + \frac{\kappa \phi_b}{\lambda_b} (\lambda_b - c) \right),
\end{align*}
\]

where HC stands for the high-cost and LC for the low-cost regimes. Furthermore, \( \alpha_{HC} \) (resp. \( \alpha_{LC} \)) is the lower boundary point of \( M_b \) in the high (resp. low) cost regime, that follows equation (2.13). It is helpful to examine the distance between the steady state distribution of VCs’ reputation, denoted by \( P_\infty \), and its perfect learning benchmark, denoted by \( P^* = (1 - p)\delta_0 + p\delta_1 \). These two probability measures assume different supports thus are

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A unit mass concentrated on \( x \) is denoted by \( \delta_x \).
not absolutely continuous with respect to each other. Therefore, I choose the total variation as a natural candidate for their distance. Let $\mathcal{B}[0, 1]$ be the Borel $\sigma$-field on the unit interval, then

$$d_{TV}(P_\infty, P^*) = \sup \{ |P_\infty(A) - P^*(A)| : A \in \mathcal{B}[0, 1] \}.$$  \hspace{1cm} (4.2)

In both regimes $P_\infty = \frac{1-p}{1-\alpha_b} \delta_{\alpha_b} + \frac{p-\alpha_b}{1-\alpha_b} \delta_{1}$, but with different $\alpha_b$’s resulted from distinct $\omega(1)$’s in (2.13). Then a simple analysis for $\alpha_b \leq p < 1$ yields

$$d_{TV}(P_\infty, P^*) = (1-p) \max \left\{ \frac{\alpha_b}{1-\alpha_b}, \frac{1}{1-\alpha_b}, 1 \right\} = \frac{1-p}{1-\alpha_b}. \hspace{1cm} (4.3)$$

A substantial result of this analysis is that a lower distance between steady state reputation measure and the perfect learning benchmark is associated to lower values of $\alpha_b$ and corresponds to higher welfare outcomes followed from (4.1). Notice that one can interpret the surplus expressions in (4.1) as the product of the extensive margin and the intensive margin, for example in the high-cost regime:

$$rS_{HC} = \frac{p-\alpha_{HC}}{1-\alpha_{HC}} \frac{\kappa \varphi_b / \lambda_b}{1 + \kappa \varphi_b / \lambda_b} \left( \frac{\lambda_b - c}{\lambda_b} \right)$$ \hspace{1cm} (4.4)

In an ideal environment the type of every investor is known to herself and to the public, thus only the high-skilled VCs with the mass of $p$ would invest and the others stay inactive, corresponding to the maximum of the extensive margin with $\alpha = 0$, and reaching the maximum surplus $S_{\text{max}}$. In figure 4 the ratio of the equilibrium surplus in the high cost regime over the maximum surplus (when the learning is perfect) is plotted as a function of $\alpha$, that is clearly decreasing, supporting the fact that a closer distance to the perfect learning benchmark is associated with smaller loss of surplus. Further in this graph, I plotted the matching value function as a function of the reputation $\pi$, that is shown to smoothly meet the horizontal axis at $\alpha$. Thus, any policy that is aimed to push down the termination point $\alpha$ toward the origin, equivalently easing up the financial costs for VCs, or encouraging higher tolerance for failure reduces the welfare gap.
5  Early/Late stage and endogenous mass of projects

In the previous sections we saw that the investment decisions made by VCs are actually constraint Pareto optimal when the mass of available projects are exogenous. However, one could envision an economy where these masses depend on the past decisions of investors, so they are endogenously determined in the equilibrium. Specifically, the choice of the matching sets analyzed in previous sections could potentially have an impact on the supply side of this economy and particularly the mass of available projects (see figure 5). Let us interpret the two types of available projects, i.e $\{a, b\}$, as early and late stage ventures. In the venture capital industry the early stage startups are usually classified as those that are early in the fund-raising cycle (round B or earlier), and the late stage ones refer to more
mature companies that passed the round $C$ fund-raising. In a broader context of innovation literature, I shall interpret the early stage projects as the ones associated to risky radical innovations with longer average time to success and the late stage projects as the safer incremental ventures with shorter average time to success. In both readings, there is an spillover from successful early stage developments to the late stage opportunities. Formally, the stationary mass of $\varphi_b$ depends on the mass of successful early stage projects. Toward this construction, suppose a fraction $\zeta$ of successful early stage projects would spill over to the rest of economy, and give rise to the creation of late stage businesses. Therefore, in any steady state outcome it must be that

$$\zeta \lambda_a m_a(1) \chi_a(1) = \kappa \varphi_b n(1).$$

(5.1)

So, conditioned on $\chi_a(1) = 1$, then $\varphi_b = \zeta \varphi_a$. Consequently, if

$$\lambda_a - c > \frac{\kappa \zeta \varphi_a (\lambda_b - c)}{r + \lambda_b + \kappa \zeta, \varphi_a},$$

(5.2)

then VCs invest in early stage startups, of which the successful ones create the late stage opportunities. This is because the opportunity cost of forgoing the option to wait for the next late stage project is not high enough to preclude the investment in the early stage companies. Therefore, in the stationary equilibrium both types of companies coexist. I call this equilibrium the *maximum surplus equilibrium*. On the other hand, when $\lambda_a \leq c$, VC firms do not invest in any company, thus the investment activity is shut down, and it is referred as *zero surplus equilibrium*. Importantly, we observe that higher search frictions – translating to lower $\kappa$ – brings down the opportunity cost of forgoing the option to wait for late stage proposals, and hence increases the likelihood of investment in early stage businesses.

In each of the above two cases, there exists a unique stationary equilibrium, however, in

---

According to NVCA 2020 yearbook based on the data provided by the PitchBook, in 2019 $80.7B is invested in the late stage startups and $52.8B in early and seed stage startups. The number of deals made in the former group was 2717 and in the latter one was 8642. Hence, both groups constitute a noticeable share of total investment activity.

In more developed countries the former equilibrium seems to be the prevailed outcome, in which sizable investments are made by the venture capital industry in both early as well as late stage companies. For example in the recent survey by Gompers et al. (2020) 62% of US institutional VC firms specialize in a particular stage, among them 36% indicated they invest in seed or early stage companies and 14% invest only in mid to late stage startups.
the intermediate case

\[ c \leq \lambda_a \leq c + \frac{\kappa \zeta \varphi_a (\lambda_b - c)}{r + \lambda_b + \kappa \zeta \varphi_a}, \]

(5.3)

there is no stationary equilibrium. If it were one, then VCs must invest in both groups of companies, i.e \( 1 \in \mathcal{M}_a \cap \mathcal{M}_b \), that is not the case because \( \lambda_a \) is not high enough. Assume initially \( \varphi_b = 0 \), then VCs only invest in early stage companies, because \( \lambda_a > c \) and there is no better option available to them. As a result of subsequent spillovers, late stage opportunities start to appear, so \( \varphi_b > 0 \). Consequently, the VCs approached by the \( b \)-companies optimally choose to invest in their ventures, thereby reducing the net investment in the early stage companies. So, the population of successful early stage startups declines, lowering \( \varphi_b \) all the way down to zero again, and the economy returns to the initial point in the cycle. This mechanism essentially calls for a nonstationary equilibrium in the intermediate region (5.3).

The planner can however intervene when the economy is trapped in the zero surplus equilibrium. In particular, to shift the equilibrium, the planner can subsidize the investment in the early stage companies by taxing the output of late stage projects. That would in turn encourage the private investors to fund early stage projects, some of which turn into successful late stage companies. Once \( m_b(1) \) reaches a critical mass, the planner can tax their output to finance the permanent subsidy of early stage investments, thereby sustaining the maximum surplus equilibrium on a balanced-budget forever. So effectively, by adopting the redistributive policy the planner internalizes the positive spillovers from early stage to late stage projects in the investment decisions made by investors.

Toward a better understanding of the constrained optimum and the source of externality in this economy, I express the planner’s constrained optimization problem below. The maximand is the expected social surplus of the economy and the constraints are the dynamical equations for the population of VCs and startups. Let \( m_q(1) \) be the mass of investors with maximum reputation connected to a \( q \)-project; \( n(1) \) the mass of unmatched startups with reputation 1; \( m_q(\pi) \) the density of matched investors to a \( q \)-project, and finally \( n(\pi) \) is the density of unmatched investors with reputation \( \pi \). All of these measures are time-dependent (even though the time index \( t \) is suppressed). Therefore, the discounted social surplus of this economy is

\[ S = \int_0^\infty e^{-rt} \left( \sum_q (\lambda_q - c) m_q(1) + \int (\lambda_q \pi - c) m_q(\pi) d\pi \right) dt. \]

(5.4)

The planner chooses the time-dependent matching indicators \( \chi_q(\pi) \) to maximize \( S \) subject
to the following law of motions for the population measures:

\[
\dot{m}_q(1) = -\lambda_q m_q(1) + \kappa \varphi_q n(1) \chi_q(1), \quad \text{for } q \in \{a, b\} \tag{5.5a}
\]

\[
\dot{n}(1) = \sum_q \lambda_q m_q(1) - \sum_q \kappa \varphi_q n(1) \chi_q(1) + \sum_q \int \lambda_q \pi m_q(\pi) d\pi \tag{5.5b}
\]

\[
\dot{m}_q(\pi) = -\lambda_q \pi m_q(\pi) + \kappa \varphi_q n(\pi) \chi_q(\pi) + \lambda_q \partial_\pi (\pi(1 - \pi) m_q(\pi)), \quad \text{for } q \in \{a, b\} \tag{5.5c}
\]

\[
\dot{n}(\pi) = -\sum_q \kappa \varphi_q n(\pi) \chi_q(\pi) \tag{5.5d}
\]

The first equation above combines the in- and out-flows from \( m_q(1) \), namely the outflow of successful ventures of type \( q \) and the inflow of the matches between type-\( q \) projects with investors of reputation 1, conditioned on the matching indicator \( \chi_q(1) \). The second law of motion accounts for the flows in and out of \( n(1) \). The first and the last term represent the inflow from successful matches whose investors now become unmatched, while the second term captures the outflow due to currently formed partnerships between investors with maximum reputation and all admissible projects. The third forward equation captures how the population of investors with reputation \( \pi \) that are matched with type-\( q \) projects evolves. The first term is the outflow of those who become successful and thus leave the group; the second term is the inflow of recently formed partnership; and the third term is the net learning inflow: summing the inflow of the group with reputation in \( (\pi, \pi + d\pi) \) who experience a decline in reputation, and the outflow of the ones leaving \( (\pi - d\pi, \pi) \). The fourth equation expresses how the density of unmatched investors with intermediate reputation declines over time. The last state constraint that should be considered in the planner’s problem is

\[
\dot{\varphi}_b = \zeta \lambda_a \left( m_a(1) + \int \pi m_a(\pi) d\pi \right) - \kappa \varphi_b \left( n(1) \chi_b(1) + \int n(\pi) \chi_b(\pi) d\pi \right) \tag{5.6}
\]

This equation relates the rate of change of the mass of available late stage projects (\( \dot{\varphi}_b \)) to the inflow originated from spillovers of successful early stage ventures and the outflow of the recent partnerships made with the members of the unmatched late stage group.

Let \( \{v_*(1, q), w_*(1), v_*(\pi, q), w_*(\pi)\} \) respectively be the co-state processes for equations (5.5a), (5.5b), (5.5c) and (5.5d); each of them shall be interpreted as the social marginal value of an additional member to its associated group. For example, \( v_*(\pi, q) \) is the social marginal value of adding one more \( (\pi, q) \) match. Also, denote the co-state process for equation (5.6) by \( \rho \). Packed up by these state equations, I solve for the social optimum of this economy by
analyzing the current value Hamiltonian in appendix A.5. It is established there that from the planner’s viewpoint:

\[ \chi^*_q(\pi) = 1 \iff v_*(\pi, q) > w_*(\pi) \]  

(5.7)

Particularly, a match between an investor with reputation \( \pi \) and a type-\( q \) project is socially optimal if the social marginal value of the match \( v_*(\pi, q) \) exceeds the social marginal value of holding reputation \( w_*(\pi) \).

Also shown in the appendix, in the steady state, where the time derivatives are zero, the following co-state equations are resulted for social contributions:

\[ rv_*(1, q) = \lambda_q - c + \lambda_q (w_*(1) - v_*(1, q)) + \rho \zeta \lambda_{a1(q=a)} \chi^*_q(1) \text{ if } \chi^*_q(1) = 1 \]

\[ rw_*(1) = \sum_q \kappa \varphi_q (v_*(1, q) - w_*(1)) - \chi^*_q(1) - \rho \kappa \varphi_b \chi^*_b(1) \]

\[ rv_*(\pi, q) = \lambda_q \pi - c + \lambda_q \pi (w_*(1) - v_*(\pi, q)) - \lambda_q \pi (1 - \pi) v'_*(\pi, q) + \rho \zeta \lambda_{a1(q=a)} \chi^*_q(\pi) \text{ if } \chi^*_q(\pi) = 1 \]

\[ rw_*(\pi) = \sum_q \kappa \varphi_q (v_*(\pi, q) - w_*(\pi)) \chi^*_q(\pi) - \rho \kappa \varphi_b \chi^*_b(\pi) \]

(5.8)

The above differential characterization of the social value functions should be juxtaposed with the private valuations of (2.3) and (2.5). In particular, the terms in the boxes precisely characterize the sources of the departure of the social from private incentives. These terms can guide us about the profile of taxes that decentralizes the social optimum. When there is spillovers from early stage to late stage businesses the above expressions suggest the following redistributive schedule:

- Cost subsidization of early stage projects.
- Taxing the output of late stage businesses.

The subsidization \( (\rho \zeta \lambda_a \pi) \) can be made either as a flow payment that depends on the current value of investor’s reputation \( (\pi) \), or equivalently (and much easier) as a one-off rebate that investors of early stage projects receive upon the success with the face value of \( \rho \zeta \). On the other hand the tax imposed on the unmatched investors is \( \rho \kappa \varphi_b \), where \( \varphi_b = \zeta \varphi_a \) in the steady state level resulted from equation (5.6).

Even though the direction of the corrective tax/subsidy seems natural, there are real world examples where the implementation amendments undo the original promise of the government intervention. For example, the Finnish Industry Investment Ltd (FII), a government owned investment agency, started its operation in 1995 with the core mandate of financing and stimulating the venture capital funds investing
In the steady state the redistributive schedule is budget neutral, so the planner runs no deficit or surplus. This is owed to the following accounting analysis:

\[
\begin{align*}
\text{total subsidy} &= \rho \zeta \lambda_a m_a(1) + \rho \zeta \lambda_a \int \pi m_a(\pi) d\pi \\
\text{total tax revenue} &= \rho \kappa \varphi_b \left( n(1) \chi_b(1) + \int n(\pi) \chi_b(\pi) d\pi \right)
\end{align*}
\] (5.9)

Since in the steady state \( \dot{\varphi}_b = 0 \), the above two sums match each other, and the redistribution is self-financing. Furthermore, in the steady state of this economy the densities should be identically equal to zero, and all the masses concentrate discretely on the boundaries. This observation hints to the condition under which intervention, namely setting \( \chi_a(1) = 1 \), is justified in steady state. Particularly, \( \chi^*_a(1) = 1 \) iff the resulting social surplus exceeds zero, which is what economy achieves when \( \chi_a(1) = 0 \) and \( \varphi_b = \chi_b(1) = 0 \). So,

\[
\chi^*_a(1) = 1 \iff \lambda_a m_a(1) + \lambda_b m_b(1) > 0.
\] (5.10)

This condition translates to

\[
\chi^*_a(1) = 1 \iff \zeta > \frac{\lambda_b (c - \lambda_a)}{\lambda_a (\lambda_b - c)}.
\] (5.11)

There is a very important message behind this derivation: the centralized intervention – in form of tax and subsidy and even setting the choice of matching sets – is justified if and only if the spillovers from early to late stage ventures is large enough.

Using this observation the first order condition for \( \rho \), i.e the shadow social value of \( \varphi_b \), is presented in the appendix, which in the steady state reduces to:

\[
r \rho = \kappa n(1) (v_*(1, b) - w_*(1) - \rho) \chi^*_b(1)
\] (5.12)

This equations confirms that that \( \rho \geq 0 \) and therefore the direction of transfers explained above is indeed correct.

Next, I explain what can be achieved in the steady state of the economy if the tax/subsidy scheme can only depend on the type of the projects and not on whether the investors are in seed and early stage startups. However, FII was also set by the government to operate profitably. In the evaluation report published in Murray and Maula (2003), it is stated that this requirement “has led the organization to seek later stage investments in order to meet the profitability target”.

The failure to correctly predict the extent of such positive spillovers doomed the sizable upfront investments that the Malaysian government made to boost the biotechnological developments in BioValley Lerner (2002).
matched or not.

**Proposition 7.** If the transition probability to late stage startups is large enough, particularly \(\lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c) < \zeta \leq 1\), then there exists a budget-balanced redistribution scheme that shifts the economy from zero surplus to maximum surplus equilibrium that depends on the type of projects and not on the matching status of their investors.

**Proof.** In the case where VCs invest in both types of companies, the total surplus is \((\lambda_a - c)m_a(1) + (\lambda_b - c)m_b(1)\) that is equal to

\[
r \cdot S = \frac{p - \alpha}{1 - \alpha} \frac{\kappa \varphi_a}{1 + \kappa \varphi_a/\lambda_a + \kappa \zeta \varphi_a/\lambda_b} \left( \frac{\lambda_a - c}{\lambda_a} + \frac{\zeta(\lambda_b - c)}{\lambda_b} \right).
\]

Therefore, the created surplus is positive iff \(\zeta > \lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c)\), and only then is it optimal to intervene. The goal of any redistribution policy should be to provide enough incentives to VCs to invest in both types of companies, so that the intensive margin – present value of the profit from a generic investment – be nonzero, and the market does not break down.

Toward this, assume the planner collects a fraction \(t_b\) of the success outcome of late stage companies, and use this revenue to subsidize the investment to early (resp. late) stage startups by a fraction \(s_a\) (resp. \(s_b\)). Budget neutrality requires

\[
m_a(1)c s_a + m_b(1)c s_b = \lambda_b m_b(1)t_b.
\]

(5.14)

Since at steady state \(\lambda_b m_b(1) = \kappa \varphi_b n(1) = \kappa \zeta \varphi_a n(1)\), then

\[
\frac{cs_a}{\zeta \lambda_a} + \frac{cs_b}{\lambda_b} = t_b.
\]

(5.15)

Further, note that the value of holding maximum reputation after redistribution when \(1 \in \mathcal{M}_a \cap \mathcal{M}_b\) is

\[
w_{ab}(1) := \frac{r^{-1} \kappa \zeta \varphi_a (\lambda_b(1 - t_b) - c(1 - s_b))(r + \lambda_a) + r^{-1} \kappa \varphi_a (\lambda_a - c(1 - s_a))(r + \lambda_b)}{(r + \lambda_a)(r + \lambda_b) + \kappa \zeta \varphi_a (r + \lambda_a) + \kappa \varphi_a (r + \lambda_b)}.
\]

(5.16)
The incentive constraints for $1 \in \mathcal{M}_a \cap \mathcal{M}_b$, resulted from proposition 4, are

$$
\lambda_a - c(1 - s_a) > r w_{ab}(1) \iff \lambda_a - c(1 - s_a) > \frac{\kappa \zeta \phi_a (\lambda_b(1 - t_b) - c(1 - s_b))}{r + \lambda_b + \kappa \zeta \phi_a},
$$
\begin{equation}
(5.17a)
\end{equation}

$$
\lambda_b(1 - t_b) - c(1 - s_b) > r w_{ab}(1) \iff \lambda_b(1 - t_b) - c(1 - s_b) > \frac{\kappa \zeta \phi_a (\lambda_a - c(1 - s_a))}{r + \lambda_a + \kappa \zeta \phi_a}.
$$
\begin{equation}
(5.17b)
\end{equation}

The planner must design $(s_a, s_b, t_b)$, subject to the budget-balanced condition (5.15) and the above incentive constraints. I define $e_b := t_b - \frac{\sigma b}{\lambda b}$ as the effective tax-rate on $b$-companies. Assume $e_b$ is small enough, so that the expected payoff from investment in $b$-startups is higher than that of $a$-startups and larger than zero, namely

$$
\lambda_b(1 - t_b) - c(1 - s_b) \geq \max \{ \lambda_a - c(1 - s_a), 0 \}.
$$
\begin{equation}
(5.18)
\end{equation}

This amounts to an upper-bound on the effective tax rate $e_b$, that automatically guarantees (5.17b):

$$
e_b \leq \min \left\{ \frac{\lambda_b - \lambda_a}{\lambda_b + \zeta \lambda_a}, \frac{\lambda_b - c}{\lambda_b} \right\} = \frac{\lambda_b - \lambda_a}{\lambda_b + \zeta \lambda_a} < 1
$$
\begin{equation}
(5.19)
\end{equation}

The middle identity holds because $\zeta > \lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c)$. The incentive constraint (5.17a) boils down to

$$
e_b > \frac{(c - \lambda_a)(r + \lambda_b) + \kappa \zeta \phi_a(\lambda_b - \lambda_a)}{\zeta(\lambda_a(r + \lambda_b) + \kappa \zeta \phi_a(\lambda_b + \zeta \lambda_a))}.
$$
\begin{equation}
(5.20)
\end{equation}

Therefore, I have to show for every $\zeta \in (\lambda_b(c - \lambda_a)/\lambda_a(\lambda_b - c), 1]$, the upper bound on the effective tax rate in (5.19) is larger than the lower bound in (5.20), so that one can always find an incentive compatible redistribution. For this note that

$$
\frac{(c - \lambda_a)(r + \lambda_b) + \kappa \zeta \phi_a(\lambda_b - \lambda_a)}{\zeta(\lambda_a(r + \lambda_b) + \kappa \zeta \phi_a(\lambda_b + \zeta \lambda_a))} < \frac{\lambda_b - \lambda_a}{\lambda_b + \zeta \lambda_a} \iff \frac{\lambda_b(c - \lambda_a)}{\lambda_a(\lambda_b - c)} < \zeta.
$$
\begin{equation}
(5.21)
\end{equation}

Therefore, one can always find a range of redistributive schemes inducing the investment in both early and late stage companies incentive compatible, while being budget balanced. Specifically, any choice in this region leaves the intensive margin of the social surplus function unaffected, because after all it is a redistribution policy. In addition, for any choice in this interval, the subsidy rate for $a$-companies is less than one, because $s_a = \zeta \lambda_a e_b/c < 1$, due to the fact that $\lambda_a < c$, $\zeta < 1$ and $e_b \leq \frac{\lambda_b - \lambda_a}{\lambda_b + \zeta \lambda_a} < 1$.

At the heart of both methods of interventions (namely equation 5.8 and the previous proposition) is the intuitive idea that subsidizing the early stage projects – financed by a tax on late stage output – prevents the market breakdown originated from investors’ failure
to internalize the spillovers from early to late stage projects. The spirit of this failure is somewhat reminiscent of the neglect of R&D positive spillovers by the private sector (Lerner (2002) and Griliches (1991)).

Lastly in this section I signify the point that the proposed interventions are only justifiable when there is large enough spillover from early to late stage projects. This was formally seen in the planner’s problem and in the proof of proposition 7. There are ample examples of failed government interventions in the venture capital industry which the central planner had made upfront investments to jump-start the investment activity but these efforts were not picked up by the private sector later on.

6 Reputation and deal flow

In this section, I aim to examine the equilibrium outcomes when there is reputational externality at play. Specifically, I ask what are the indirect impacts of a reputable actor on the remaining body of investors? This question stands in a contrast to what has been so far studied about the reputation effects in the venture capital industry.

Particularly, there is a considerable body of research highlighting the individual benefits associated with higher reputation. The findings include the lower pay-for-performance for smaller and younger firms (associated with the theory of grandstanding Gompers (1996)) aimed at establishing reputation and enjoying its subsequent benefits; Or how VCs with higher reputation acquire startup equity at a discount Hsu (2004). Also noted are the preferential access to deal flow (Nanda et al. (2020)) and the ability to become more central in the VCs’ syndication network and thereby receiving a larger set of proposals (Sorenson and Stuart (2001)). However, less is known about the social returns to reputation. One may except inefficiencies would arise in an economy in which highly reputable VCs slow down the deal flow of less reputable ones, and hence hindering the learning and investing opportunities of the latter group, and eventually the overall economy. This force would have not been a concern if there was a price for reputation and a centralized market in which startups could partner with VCs. Yet, the predominant feature of this economy is the time cost of the search that underlies the VCs’ investment decisions, and the dispersed investment opportunities.

The book by Lerner (2012) reviewed many of these examples.
6.1 Equilibrium outcome with long-lived investors

To capture the aforementioned interaction effect, I would propose a different matching technology. Thus far, the matching function was assumed *uniformly* quadratic. That is over every period \(dt\) the total mass of proposals from entrepreneurs to investors is \(\kappa(\varphi_a + \varphi_b)dt\), and it is uniformly distributed among the unit mass of VCs. Holding the total rate of proposals constant, now I assume this flow is not uniformly distributed among VCs, rather it contacts more (resp. less) reputable VCs with higher (resp. lower) probability, according to the *reputation-weight* function \(\psi(\cdot)\). Specifically, let \(\pi_\infty\) be the stationary distribution of VCs’ reputation. Then the rate with which \(q\)-companies meet a VC with reputation \(\pi\) is

\[
\kappa \varphi_q \frac{\psi(\pi)}{\mu},
\]

where \(\mu := E[\psi(\pi_\infty)]\). \(^{(6.1)}\)

I assume \(\psi \geq 0, \psi' \geq 0, \psi(0) = 0\) and \(\psi(1) = 1\). One might expect that any hope to prove a uniqueness theorem such as the one in theorem 5 without having a much more restrictive assumptions on \(\psi(\cdot)\) is doomed to fail. This is mainly because the analogue of proposition 4 – in which we prove the convexity of matching sets – for the general reputation weight function is very complicated and requires making a collection of assumptions on \(\psi(\cdot)\) in conjunction with other primitives. However, to a large extent this analysis is futile in this context, because alternatively I propose an equilibrium that exists for every \(\psi(\cdot)\) satisfying the above minimal conditions. Consequently, we can perform the comparative statics on this equilibrium with respect to the choice of \(\psi(\cdot)\).

Inspired by the analysis in section 2, I conjecture that there exists an equilibrium featuring \(\mathcal{M}_b = (\alpha_e, 1], \mathcal{M}_a \subset \mathcal{M}_b\) and \(1 \in \mathcal{M}_a\) iff

\[
\lambda_a - c > \frac{\kappa \varphi_b (\lambda_b - c)}{\mu (r + \lambda_b) + \kappa \varphi_b}. \quad (6.2)
\]

Further, in this equilibrium the value of holding the maximum reputation is

\[
w(1) = \max_x \left\{ \frac{r^{-1} \kappa [\varphi_b (\lambda_b - c) (r + \lambda_a) \chi_b(1) + \varphi_a (\lambda_a - c) (r + \lambda_b) \chi_a(1)]}{(r + \lambda_a) (r + \lambda_b) \mu + \kappa \varphi_b (r + \lambda_a) \chi_b(1) + \kappa \varphi_a (r + \lambda_b) \chi_a(1)} \right\}, \quad (6.3)
\]

Notice that \(\mu\) is the *steady-state* average reputation weight, and is not the current population average of reputation weights, i.e \(\int_0^1 \psi(\pi_\alpha) \, dt\). This assumption simplifies the equilibrium analysis, particularly by letting us to focus on the time-independent termination policies, i.e constant \(\alpha\) over time.
and $\alpha_e$ is the fixed-point of the following system:

$$
\mu = \frac{1 - p}{1 - \alpha} \psi(\alpha) + \frac{p - \alpha}{1 - \alpha} \tag{6.4a}
$$

$$
\alpha = \frac{c}{\lambda_b (1 + w(1))} \tag{6.4b}
$$

Relation (6.4a) is owed to the presence of $\frac{1 - p}{1 - \alpha}$ VCs with reputation $\alpha$ and the remaining $\frac{p - \alpha}{1 - \alpha}$ with reputation one in the steady state. And equation (6.4b) simply expresses the endogenous termination point in line with the analysis offered for (2.13). I refer to any equilibrium with the above features as normal equilibrium.

**Proposition 8.** In the described economy with reputational externality,

(i) there always exists a normal equilibrium with $\alpha_e < p$.

(ii) If $\psi'' \leq 0$ the normal equilibria are Pareto ranked. Further, the $\alpha_e$ for the most (least) preferred equilibrium is increasing with respect to the pointwise order on $\psi$.

Part (i) ensures the existence of the normal equilibrium under the new choice of the matching function that exhibits the reputational externality. In light of that, we can safely claim that the sort of matching sets depicted in figure 3 are applicable in this case as well. Specifically, the normal equilibria requires the matching sets to be connected and hence the outcome of learning in the economy at the steady state can be characterized by examining the masses at the endpoints, i.e $\pi \in \{1, \alpha_e\}$.

Emboldened by the existence of normal equilibria, the analogue of the results based on proposition 4 would apply in this section too, with the change of $\kappa \varphi_q$ to $\kappa \varphi_q / \mu$ in all expressions. Specifically, when it comes to cost regime determination, the characterization (2.12) changes to (6.2). In a meaningful contrast with the baseline model – where the reputational externality was absent in the matching function – the investors’ equilibrium response to whether invest on $a$-projects depends on the average reputation score ($\mu$) of the whole body of investors. Specifically, any increase in the equilibrium value of $\mu$ lowers the opportunity cost of forgoing the option to wait for $b$-projects, that in turn relaxes the constraint for investing in $a$-projects. Therefore, softening the extent of reputational externalities would encourage investors toward the early stage projects. To sharpen the meaning behind softening the reputational externality, I examine the effect of the choice of $\psi$ as a parameter picked from the following family of admissible functions:

$$
\Psi := \{ \psi : [0, 1] \to [0, 1] | \psi(0) = 0, \psi(1) = 1, \psi' \geq 0, \psi'' \leq 0 \}, \tag{6.5}
$$
Endow $\Psi$ with the pointwise order, that is $\psi_2 \succeq \psi_1$ iff $\psi_2(x) \geq \psi_1(x) \ \forall x \in [0,1]$ (see figure 6). Inspired by this figure, I say $\psi_2$ is softer than $\psi_1$, because the marginal return to a higher reputation in $\psi_2$ is smaller than $\psi_1$. In part (ii) of the previous proposition, it is shown that the equilibrium termination point $\alpha_e$ is increasing w.r.t to $\succeq$ on $\Psi$. Therefore, softening the extent of reputational externality (i.e increasing $\psi$ in a pointwise manner), reduces the investors’ patience (i.e increases the equilibrium $\alpha_e$), by lowering the equilibrium value of reputation building (i.e $w(1)$), and thereby relaxing the constraint for investing on $a$-projects in equation (6.2). The following line summarizes the result of this comparative static exercise:

$$ \psi \uparrow \Rightarrow w(1) \downarrow, \ \mu_e \uparrow \text{ and } \alpha_e \uparrow \quad (6.6) $$

Because of the reputational externality, one would expect under-learning in the equilibrium outcome relative to the social optimum. That is the reputable group of investors receive a higher than socially optimal share of investment proposals, leaving the lesser known group with fewer options, thereby lowering their reservation value $w$.

The comparison of the steady state equilibrium surplus with the steady state social optimum in the current environment of long-lived agents ignores the previous costs born by investors on the investment path (starting from $p$ and ending at $\alpha_e$). Particularly, the steady state social surplus is maximized at $\alpha = 0$, because it fails to take into account the cost of pushing $\alpha$ down to zero. This is owed to the fact that in the steady state there will
be no investors with reputation in \((\alpha_e, p]\). Therefore, in the next subsection, I will allow for exogenous birth and death of investors to obtain a non-degenerate stationary economy, justifying the comparison of the steady state equilibrium outcome with the steady state social optimum, by the means of having a continuous distribution of investors on \((\alpha_e, p]\). This tweak helps us to understand the spirit of the reputational externality and the extent to which the decentralized outcome under-appreciates the gains from more patience.

### 6.2 Short-lived investors

The nature of reputational externality can be easily described if we focus only on one group of projects, say the \(b\)-startups and henceforth in this section I drop the \(b\)-index from variables. Since the focus of the forthcoming analysis is the stationary distribution of VCs’ reputation and its impact on the investment pattern, and not the spillovers between different types of projects, this assumption is largely innocuous.

The investors are short-lived. Specifically, they leave the economy exogenously at the rate of \(\delta\), and are born with the same rate and the initial reputation \(p\). The matching function is quadratic and exhibits reputational externality normalized by the steady state reputation score \(\mu = E[\psi(\pi_\infty)]\). I assume minimal structure on \(\psi\) by letting it be only increasing and concave, and fixing \(\psi(1) = 1\). I conjecture (and prove) that there exists a symmetric stationary equilibrium in which all investors terminate their matches at a common \(\alpha\). In light of this conjecture, denote the cross-sectional density function of the matched VCs by \(m(\pi)\) supported on \([\alpha, p]\). Let \(m(1)\) and \(n(1)\) be the discrete measures of the matched and unmatched VCs with maximum reputation, respectively, and finally \(n(\alpha)\) and \(n(p)\) are the discrete measures of unmatched VCs at \(\alpha\) and \(p\). Figure 7 plots all pieces of the cross-sectional steady state distribution of investors’ reputations.

The inflow outflow equations at the discrete masses are

\[
\dot{m}(1) = -\lambda m(1) + \kappa \varphi \frac{n(1)}{\mu} - \delta m(1) \quad (6.7a)
\]

\[
\dot{n}(1) = \lambda m(1) - \kappa \varphi \frac{n(1)}{\mu} - \delta n(1) + \int_\alpha^p \lambda \pi m(\pi) d\pi \quad (6.7b)
\]

\[
\dot{n}(p) = -\kappa \varphi \frac{\psi(p)}{\mu} n(p) - \delta n(p) + \delta \quad (6.7c)
\]

Notice that \(n(\alpha)\) is determined through population conditions such as the conservation of reputation weight.

Recall from the previous section that taking \(\mu\) as the steady state average of reputation weights supports a time-invariant termination point \(\alpha\) in the equilibrium.
first and second moment. The forward equation for $m(\pi)$ is

$$\dot{m}(\pi) = -\lambda \pi m(\pi) + \lambda \partial_\pi (\pi (1-\pi) m(\pi)) - \delta m(\pi). \quad (6.8)$$

The first component in the rhs is the outflow from $m(\pi)$ (due to the recent success events) to $n(1)$. The second term captures the net learning effect, by factoring the inflow of investors whose reputation is in $(\pi, \pi+d\pi)$ and thus falling due to the lack of success and the outflow of the unsuccessful group with reputation in $(\pi-d\pi, \pi)$. Finally, the third term is associated to the exogenous departures. In the steady state $\dot{m}(\pi) = 0$ thus raising a differential equation for the density function whose solution is

$$m(\pi) = m(\alpha) \left( \frac{\pi}{\alpha} \right)^{\delta/\lambda-1} \left( \frac{1-\pi}{1-\alpha} \right)^{-\delta/\lambda+2}, \quad \forall \pi \in [\alpha, p]. \quad (6.9)$$

The group of VCs with minimum reputation at $\pi = \alpha$ are subject to two flows: the inflow from the matched ones in $(\alpha, p]$ and the outflow due to the exogenous exits at the rate of $\delta n(\alpha)$. Therefore, in the steady state it must be that the inflow equals $\delta n(\alpha)$.

Lastly, the net inflow to the matched VCs on the region $[\alpha, p]$ must match the net outflow.
in the steady state, that is
\[ \frac{\psi(p)}{\mu} n(p) = \lambda \int_\alpha^p \pi m(\pi) d\pi + \delta \int_\alpha^p m(\pi) d\pi + \delta n(\alpha). \] (6.10)

**Lemma 9.** In the steady state of the above economy,
\[ \int_\alpha^p m(\pi) d\pi = \frac{\kappa \varphi \psi(p)/\mu}{\delta + \kappa \varphi \psi(p)/\mu} \frac{p - \alpha}{\mu} \left( \Upsilon_1(\alpha) - \frac{\lambda}{\delta + \lambda} \Upsilon_2(\alpha) \right), \] (6.11a)
\[ \int_\alpha^p \pi m(\pi) d\pi = \frac{\kappa \varphi \psi(p)/\mu}{\delta + \kappa \varphi \psi(p)/\mu} \frac{\delta}{\mu} \frac{p - \alpha}{\mu} \Upsilon_2(\alpha), \] (6.11b)
\[ m(1) = \frac{\kappa \varphi \psi(p)/\mu}{\delta + \kappa \varphi \psi(p)/\mu} \frac{\lambda}{\mu} \Upsilon_2(\alpha) - \frac{\kappa \varphi / \mu}{\delta + \kappa \varphi / \mu} \Upsilon_1(\alpha), \] (6.11c)
\[ n(\alpha) = \frac{\kappa \varphi \psi(p)/\mu}{\delta + \kappa \varphi \psi(p)/\mu} \frac{\Upsilon_2(\alpha) - \alpha \Upsilon_1(\alpha)}{\mu} \] (6.11d)

where
\[ \Upsilon_i(\alpha) := \left( \frac{p}{\alpha} \right)^{\delta / \lambda - 1}\left( \frac{1 - p}{1 - \alpha} \right)^{-(\delta / \lambda + 2)} p^i(1 - p) - \alpha^i(1 - \alpha), \quad \text{for } i \in \{1, 2\}. \] (6.12)

Given the results found in this lemma one can examine the limits when the VCs become long-lived agents, that is as \( \delta \to 0 \). It is easy to verify that for both \( i \in \{1, 2\} \):
\[ \Upsilon_i(\alpha) \to \alpha(1 - \alpha) \frac{p - \alpha}{1 - p} \] (6.13)
And accordingly \( \int_\alpha^p m(\pi) d\pi \to 0 \), \( \int_\alpha^p \pi m(\pi) d\pi \to 0 \), \( m(1) \to \frac{\kappa \varphi / \mu}{\lambda + \kappa \varphi / \mu} \frac{p - \alpha}{1 - \alpha} \), and \( n(\alpha) \to \frac{1 - p}{1 - \alpha} \) as \( \delta \to 0 \); confirming the previous results on the economy with long-lived investors.

Toward the equilibrium analysis, each investor stipulates the population average for \( \psi \), say \( \mu \), and accordingly specifies the maximum attainable reputation value via the mapping \( W : [0, 1] \to \mathbb{R}_+ \):
\[ W(\mu) := \frac{(r + \delta)^{-1} \kappa \varphi / \mu}{r + \delta + \lambda + \kappa \varphi / \mu} (\lambda - c) \] (6.14)
Then, following the Bellman equation on the continuation region induced by \( w(1) = W(\mu) \), namely
\[ rv(\pi) := \lambda - c + \lambda (w(1) - v(\pi)) - \lambda \pi (1 - \pi) v'(\pi) - \delta v(\pi), \] (6.15)
the investor terminates the project at \( \alpha = A(w(1)) \), where \( A : \mathbb{R}_+ \to [0, 1] \) and

\[
A(w) := \frac{c}{\lambda(1 + w)}. \tag{6.16}
\]

In the symmetric stationary equilibrium the initial stipulation about \( \mu \) is self-fulfilling that is \( \mu = M(\mu, A \circ W(\mu)) \), where \( M : [0, 1]^2 \to \mathbb{R}_+ \) returns the population average of reputation weights:

\[
M(\mu, \alpha) = E[\psi(\pi_{\infty})] = m(1) + n(1) + \psi(p)n(p) + \int_{\alpha}^{p} \psi(\pi)m(\pi)d\pi + \psi(\alpha)n(\alpha) \tag{6.17}
\]

**Definition 10** (Symmetric stationary equilibrium). The symmetric stationary equilibrium in this economy with reputational externality is the set of all fixed-points of the mapping \( M(\cdot, \alpha) \) on the unit interval; A generic member is denoted by \( \mu_e \). Associated to the equilibrium outcome \( \mu_e \) is the equilibrium termination point \( \alpha_e = A \circ W(\mu_e) \).

In the appendix A.8, I show that an increase in \( \alpha \) or \( \mu \), holding the other variable constant, positively shifts the steady state distribution of \( \pi_{\infty} \) in the sense of *second-order stochastic dominance*. So, assuming a concave increasing form for \( \psi(\cdot) \) one can deduce that \( M(\mu, \alpha) \) is an increasing function in each argument. In addition to that, the composition map \( A \circ W \) is increasing, therefore the mapping \( \mu \mapsto M(\mu, A \circ W(\mu)) \) is a continuous increasing function from the unit interval to itself. Hence, a fixed-point \( \mu_e \) and \( \alpha_e = A \circ W(\mu_e) \) exist, establishing the existence of a symmetric stationary equilibrium.

To contrast the equilibrium outcome with the socially optimal choice, I express the steady state flow surplus of the economy in terms of the measures found in lemma 9:

\[
rS(\mu, \alpha) = (\lambda - c)m(1) + \int_{\alpha}^{p} (\lambda \pi - c)m(\pi)d\pi
= \frac{\kappa \phi(\pi)/\mu}{\delta + \kappa \phi(\pi)/\mu} \frac{(p - \alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha) - \alpha \Upsilon_1(\alpha)} \times \left\{ \lambda \left( \frac{\delta}{\delta + \lambda} + \frac{\kappa \phi/\mu}{\delta + \lambda + \kappa \phi/\mu \delta + \lambda} \right) - c \left( \frac{\Upsilon_1(\alpha)}{\Upsilon_2(\alpha)} - \frac{\lambda}{\delta + \lambda} + \frac{\kappa \phi/\mu}{\delta + \lambda + \kappa \phi/\mu \delta + \lambda} \right) \right\} \tag{6.18}
\]

A benevolent social planner selects an \( \alpha \) so that jointly with its induced \( \mu \), that is the fixed-point of \( M(\cdot, \alpha) \), maximize the social surplus \( S(\mu, \alpha) \).

---

It is clearly continuous on \((0, 1]\), and it is made continuous at \( \mu = 0 \) by letting \( W(0) := \lim_{\mu \to 0} W(\mu) \) and \( M(0, \alpha) := \lim_{\mu \to 0} M(\mu, \alpha) \), where both limits exist in light of the expression (6.14) and lemma 9.
**Definition 11** (Planner’s problem). The planner’s problem is

\[
\max S(\mu, \alpha) \text{ subject to } \mu = M(\mu, \alpha)
\]  

(6.19)

Remember the externality failed to be internalized in the investors’ decision is originated from the impact of their choices on \( \mu \). Therefore, it is essential to incorporate \( \mu = M(\mu, \alpha) \) as the constraint of the planner’s problem.

Next proposition explains why the equilibrium outcome is not socially efficient, and highlights the direction along which the social surplus increases.

**Proposition 12.** Every symmetric stationary equilibrium of the economy with reputational externality is not constrained-efficient. In particular, a local reduction in the termination point \( \alpha_e \) increases the social surplus.

**Proof.** Every symmetric equilibrium is characterized by its associated pair of termination policy \( \alpha_e \) and the population average of reputation weights \( \mu_e \), in which \( \alpha_e = A \circ W(\mu_e) \) and \( \mu_e = M(\mu_e, \alpha_e) \). It is further a stable equilibrium if \( \partial_\mu M(\mu_e, \alpha_e) < 1 \). From the expression for the social surplus in (6.18) one can see that \( S \) is decreasing in \( \mu \), therefore, if \( M(\cdot, \alpha) \) has multiple fixed-points for a given \( \alpha \) the one with the smallest \( \mu \) is the most efficient one. Furthermore, this equilibrium (with the smallest \( \mu \)) is stable because \( M(0, \alpha) > 0 \), and \( M(\cdot, \alpha) \) *downcrosses* the 45-degree line in its first intersection.

Toward proving the constrained inefficiency, I employ a variational approach in the neighborhood of \( \alpha_e \). Suppose the economy is in a stable pair \((\alpha_e, \mu_e)\), and the planner moves \( \alpha_e \) by \( \Delta \alpha \). The new smallest fixed-point \( \mu_e + \Delta \mu \) satisfies

\[
\mu_e + \Delta \mu = M(\mu_e + \Delta \mu, \alpha_e + \Delta \alpha) \approx M(\mu_e, \alpha_e) + \partial_\mu M \Delta \mu + \partial_\alpha M \Delta \alpha,
\]  

(6.20)
hence \( \Delta \mu = \frac{\partial_\alpha M}{1 - \partial_\mu M} \Delta \alpha \). Consequently, the change in the social surplus function would be

\[
r \Delta S = r \left( \frac{\partial_\alpha M}{1 - \partial_\mu M} \partial_\mu S + \partial_\alpha S \right) \Delta \alpha.
\]  

(6.21)

Note that in every stable fixed-point of \( M(\cdot, \alpha_e) \), \( \frac{\partial_\alpha M}{1 - \partial_\mu M} > 0 \), because \( M \) is shown to be increasing in \( \alpha \) and due to the stability \( \partial_\mu M < 1 \). Further, \( \partial_\mu S < 0 \) as can readily be verified from (6.18). Therefore, lowering \( \alpha_e \), i.e \( \Delta \alpha < 0 \), leads to a strict improvement in the social surplus.
surplus if $\partial_{\alpha}S < 0$. Relying on (6.18) and applying some rearrangements lead to

$$r \partial_{\alpha}S(\mu_e, \alpha_e) = (\lambda - c) \partial_{\alpha}m(1) - (\lambda \alpha - c)m(\alpha)$$

$$= -\frac{\kappa \varphi \psi(p)/\mu_e}{\delta + \kappa \varphi \psi(p)/\mu_e (1 - \alpha_e)^2} \left( \frac{p}{1 - p} \right)^{-\delta/\lambda} \left( \frac{\alpha_e}{1 - \alpha_e} \right)^{\delta/\lambda} \times$$

$$\left[ \frac{\delta (\lambda \alpha_e - c)}{\lambda \alpha_e} + \frac{(\lambda - c) \kappa \varphi / \mu_e}{\delta + \lambda + \kappa \varphi / \mu_e} \right].$$

Therefore, the sign of $\partial_{\alpha}S(\mu_e, \alpha_e)$ is opposite of the sign of the expression in the bracket. Recalling that in the equilibrium $\alpha_e = A \circ W(\mu_e)$, so

$$\frac{\delta (\lambda \alpha_e - c)}{\lambda \alpha_e} + \frac{(\lambda - c) \kappa \varphi / \mu_e}{\delta + \lambda + \kappa \varphi / \mu_e} = -\delta W(\mu_e) + \frac{(\lambda - c) \kappa \varphi / \mu_e}{\delta + \lambda + \kappa \varphi / \mu_e}$$

$$= -\delta W(\mu_e) + \delta \lim_{r \to 0} W(\mu_e) \geq 0,$$

where the last inequality holds because $W(\mu_e)$ is decreasing in $r$. This concludes that $\partial_{\alpha}S(\mu_e, \alpha_e) < 0$, and hence a small reduction of equilibrium $\alpha_e$ leads to a strict improvement of the social surplus function.

Figure 8: Social surplus with reputational externality

Figure 8 is the result of a simulation that plots the social surplus as a function of $\alpha$, while implicitly satisfying $\mu = M(\mu, \alpha)$ at every $\alpha \in [0, p]$. As it is expressed in this plot, the equilibrium termination point $\alpha_e$ is greater than the socially optimal point $\alpha_s$. Hence, the
equilibrium outcome is associated with early termination of projects, and predicts a lower tolerance for failure than what is socially efficient.

7 Conclusion

I study the decentralized outcome of a dynamic economy populated by venture capitalists with unknown abilities and projects with observable qualities, where individuals randomly meet each other subject to search frictions. Since the venture capitalists fund their portfolio startups, the path to build a reputation is going to be costly for them. Therefore, the combination of costly learning and search frictions create a group of investors with high ability yet low reputation who rationally choose to stop investing (that is referred to as dormant investors in the paper). In addition to this, the equilibrium shape of the matching sets between investors and startups rationalize a number of empirical findings in other papers, such as the relation between the tolerance for failure and investors’ reputation as well as the prevalence of the investment approach, “spray and pray”, as a consequence of cost reducing positive technological shocks.

I extend the baseline model to capture two sources of market failure: missing to internalize the innovation spillovers on the projects’ side, and under-investment as a result of the reputational externality. In the former case, when there is positive spillovers from successful early stage projects to late stage businesses and the institutions are weak to protect the property rights and intangible assets of small young firms, the decentralized outcome of the economy could feature a complete market breakdown, caused by the under-investment of venture capitalists in early stage businesses and consequently ending up with sub-optimal levels of late stage companies and social surplus. Importantly, I show there are regions where higher search frictions could save the market from breakdown, as it reduces the opportunity cost of investing in early stage startups. In the latter case, when the deal flow of a single VC is inversely impacted by the reputation of other investors, the decentralized outcome of the economy features an inefficiently small size of high ability active investors and early termination of projects. A comparative static analysis on the equilibrium outcome suggests softening the extent of reputational externality has two distinct impacts: (i) Overall, VCs become less patient, and the proportion of high ability active investors falls; (ii) The equilibrium value of reputation building falls, thereby relaxing the constraint for investing in early stage startups.

As a possible future step, one could extend the introduced model of this paper to an
economy where there is two-sided incomplete information and hence two-sided learning, that is the projects’ types as well as the investors’ types are unknown. This is a challenging question because now both sides of the economy will have long-run reputational concerns. It naturally finds its applications in other contexts that feature two sided learning: for example in the labor market, where employers and employees jointly learn about their type as well as their partner’s; Or in the educational systems, where there are incomplete information about the qualities of schools as well as students.

A Proofs

A.1 Proof of lemma 3

Suppose both matching value functions, i.e \( v(\cdot, a) \) and \( v(\cdot, b) \), are increasing in \( \pi \). Then, the representation (2.9) implies that \( w(\cdot) \) should be increasing in \( \pi \) as well. Conversely, assume \( w(\cdot) \) is increasing in \( \pi \), and hence almost everywhere differentiable on \([0, 1]\), and recall that \( v(\cdot, q) \) is the solution to the optimal stopping time problem (2.4). In that \( \tau \) is the stopping time adapted to all possible future information. However, note that no information is released until the breakthrough time \( \sigma \), hence \( \tau \) only uses the current information. This means that I can restrict the optimization space to the set of all deterministic times:

\[
\begin{align*}
v(\pi, q) &= \sup_{\tau \in \mathbb{R}_+} V(\pi, q; \tau) \\
V(\pi, q; \tau) &:= \int_0^{\tau} \left[ r^{-1} c (e^{-rt} - 1) + e^{-rt} (1 + w(1)) \right] \lambda q e^{-\lambda q t} dt \\
&\quad + (1 - \pi + \pi e^{-\lambda q \tau}) \left[ r^{-1} c (e^{-r\tau} - 1) + e^{-r\tau} w(\pi_\tau) \right].
\end{align*}
\]  

(A.1)

Since \( w \) is almost everywhere differentiable, then \( V(\cdot, q; \tau) \) inherits this property too. Let us now define \( \frac{\partial V}{\partial \pi}(\pi, q; \tau) := I_1 + I_2 + I_3 \), where

\[
\begin{align*}
I_1 &:= r^{-1} c \left[ \frac{\lambda q}{r + \lambda q} \left( 1 - e^{-(r+\lambda q)\tau} \right) - e^{-r\tau} (1 - e^{-\lambda q \tau}) \right], \\
I_2 &:= \frac{(1 + w(1)) \lambda q}{r + \lambda q} \left( 1 - e^{-(r+\lambda q)\tau} \right) - (1 - e^{-\lambda q \tau}) e^{-r\tau} w(\pi_\tau), \\
I_3 &:= e^{-r\tau} (1 - \pi + \pi e^{-\lambda q \tau}) w'(\pi_\tau) \frac{\partial \pi_\tau}{\partial \pi}.
\end{align*}
\]  

(A.2)

This is due the seminal Lebesgue theorem; see chapter 6 of Royden and Fitzpatrick (1988).
The expression for $I_1$ is zero when $\tau = 0$, and has positive derivative w.r.t $\tau$, therefore, it is non-negative for all $\tau \geq 0$. The third term $I_3$ is obviously non-negative, because $w$ is assumed increasing and due to the Bayes law $\partial \pi_\tau / \partial \pi > 0$. In regard to the second term:

$$I_2 \geq \frac{(1 + w(1)) \lambda_q}{r + \lambda_q} \left( 1 - e^{-(r + \lambda_q)\tau} \right) - \left( 1 - e^{-\lambda_q\tau} \right) e^{-r\tau} w(1)$$

$$\geq w(1) \left[ \frac{\lambda_q}{\lambda_q + r} \left( 1 - e^{-(r + \lambda_q)\tau} \right) - e^{-r\tau} \left( 1 - e^{-\lambda_q\tau} \right) \right] \quad (A.3)$$

The term in the bracket above is increasing in $\tau$ and equals zero at $\tau = 0$, therefore, it is always non-negative. To sum, $\partial V / \partial \pi \geq 0$ almost everywhere, and therefore $V$ becomes increasing in $\pi$. Since $v(\pi, q) = \sup \tau V(\pi, q; \tau)$, the matching value function $v(\cdot, q)$ must be increasing too. \hfill \Box

### A.2 Proof of proposition 4

**Proof of part (i):** At $\pi = 1$ the following fixed-point system falls out of (2.9) and the rearranged version of (2.5):

$$w(1) = \max_x \left\{ \frac{r^{-1} \kappa [v(1,a)\varphi_a \chi_a(1) + v(1,b)\varphi_b \chi_b(1)]}{1 + r^{-1} \kappa [\varphi_a \chi_a(1) + \varphi_b \chi_b(1)]} \right\} \quad (A.4a)$$

$$v(1,q) = \max \left\{ w(1), \frac{\lambda_q}{r + \lambda_q} - \frac{\lambda_q}{r + \lambda_q} w(1) \right\} \quad \text{for } q \in \{a,b\} \quad (A.4b)$$

From (A.4b) it follows that

$$\chi_a(1) = 1 \iff rw(1) < \lambda_a - c, \quad (A.5a)$$

$$\chi_b(1) = 1 \iff rw(1) < \lambda_b - c. \quad (A.5b)$$

So there are three cases that could possibly arise from (A.5):

(a) $1 \notin \mathcal{M}_b \cup \mathcal{M}_a \Rightarrow w(1) = 0$, yet this never happens because $\lambda_b > c$ implies $v(1,b) > 0$ and hence $w(1) > 0$.

(b) $1 \in \mathcal{M}_b \cap \mathcal{M}_a$ so

$$w(1) = \frac{r^{-1} \kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b} \quad (A.6)$$

The pair $v(1,a) = w(1)$ and $v(1,b) = (1 + r/\kappa \varphi_b) w(1)$ satisfy (and is the only solution of) the fixed-point system (A.4) if $\lambda_a - c \leq \frac{\kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b}$.
\(1 \in \mathcal{M}_b \cap \mathcal{M}_a\) so
\[
\begin{align*}
    w(1) = \frac{r^{-1} \kappa \varphi_b (\lambda_b - c) (r + \lambda_a) + r^{-1} \kappa \varphi_a (\lambda_a - c) (r + \lambda_b)}{(r + \lambda_a) (r + \lambda_b) + \kappa \varphi_b (r + \lambda_a) + \kappa \varphi_a (r + \lambda_b)}. \quad (A.7)
\end{align*}
\]

If \(\lambda_a - c > \frac{\kappa \varphi_b (\lambda_b - c)}{r + \lambda_b + \kappa \varphi_b}\) the above \(w(1)\) satisfies (A.5). Moreover, the obtained \(v(1,a)\) and \(v(1,b)\) from (A.4b) once replaced as the optimization input in the \textit{rhs} of (A.4a) confirms the \(w(1)\) in (A.7), thereby closing the equilibrium loop. \(\|\)

\textbf{Proof of part (ii): } In the sequel I use the symbol \(\partial A\) to denote the lower boundary of the subset \(A \subset [0,1]\). To establish the convexity of \(\mathcal{M}_b\), I first derive a useful identity for any strictly positive point \(x \in \mathcal{M}_a \cap \partial \mathcal{M}_b\). Since \(x\) is a lower-boundary point for \(\mathcal{M}_b\), then a generic VC finds it optimal to terminate the funding when \(\pi\) approaches down to \(x\). Importantly, at this point the principles of continuous and smooth fit (Dixit (2013)) must hold. The VC’s outside option just below \(x\) is equal to \(w(x)\) that is supported by the option value of meeting an \(a\)-type startup because \(x \in \mathcal{M}_a\), so
\[
\begin{align*}
    v(x,b) = w(x) = \frac{\kappa \varphi_a}{r + \kappa \varphi_a} v(x,a) \quad \text{and} \quad v'(x,b) = w'(x) = \frac{\kappa \varphi_a}{r + \kappa \varphi_a} v'(x,a). \quad (A.8)
\end{align*}
\]

Now let \(\Omega(x,q) := -c + \lambda_q x (1 + w(1))\) and \(\Gamma(x,q) := r + \lambda_q x\). Then, employing the HJB equations on the continuation region leads to
\[
\begin{align*}
    \frac{v'(x,b)}{v'(x,a)} = \frac{\lambda_a}{\lambda_b} \frac{\Omega(x,b) - \Gamma(x,b) v(x,b)}{\Omega(x,a) - \Gamma(x,a) v(x,a)}. \quad (A.9)
\end{align*}
\]

The previous two systems of equations give rise to
\[
\begin{align*}
    \frac{\kappa \varphi_a}{r + \kappa \varphi_a} \left( \frac{\lambda_b}{\lambda_a} \Gamma(x,a) - \Gamma(x,b) \right) v(x,a) = \frac{\kappa \varphi_a}{r + \kappa \varphi_a} \frac{\lambda_b}{\lambda_a} \Omega(x,a) - \Omega(x,b) \quad (A.10a)
    \Rightarrow \frac{\kappa \varphi_a}{r + \kappa \varphi_a} \left( \frac{\lambda_b}{\lambda_a} - 1 \right) rv(x,a) = -c \left( \frac{\kappa \varphi_a}{r + \kappa \varphi_a} \frac{\lambda_b}{\lambda_a} - 1 \right) - \frac{rx \lambda_b (1 + w(1))}{r + \kappa \varphi_a} \quad (A.10b)
\end{align*}
\]

Now assume to the contrary that \(\mathcal{M}_b\) is not connected, hence, it contains at least two separate open sets, say \((x_0, x_1)\) and \((x_2, x_3)\). This implies that \([x_1, x_2] \subset \mathcal{M}_a\), because otherwise \(w\) assumes zero at some point in this interval which violates the monotonicity of \(w\). Therefore, \(x_2 \in \mathcal{M}_a \cap \partial \mathcal{M}_b\), and (A.10b) holds at \(x_2\). I claim that \(x_0 \in \mathcal{M}_a \cap \partial \mathcal{M}_b\) too, because otherwise \(x_0\) would be the lower boundary point at which \(v(\cdot,b)\) smoothly meets the zero
function, hence applying continuous and smooth fit to equation (2.10) yields

\[ x_0 = \frac{c}{\lambda_b (1 + w(1))}. \]  

This expression for \( x_0 \) leads to an upper-bound for \( v(x_2, a) \) using (A.10b):

\[
\frac{\kappa \varphi_a}{r + \kappa \varphi_a} \left( \frac{\lambda_b}{\lambda_a} - 1 \right) rv(x_2, a) \leq -c \left( \frac{\kappa \varphi_a \lambda_b}{r + \kappa \varphi_a} \lambda_a - 1 \right) - \frac{r x_0 \lambda_b (1 + w(1))}{r + \kappa \varphi_a}
\]

\[ = \frac{c \kappa \varphi_a}{r + \kappa \varphi_a} \left( 1 - \frac{\lambda_b}{\lambda_a} \right) < 0. \]  

This means that \( v(x_2, a) < 0 \), hence a contradiction results. Therefore, \( x_0 \) and \( x_2 \) both belong to \( \mathcal{M}_a \cap \partial \mathcal{M}_b \). One can now apply (A.10b) at these two points and subtract their corresponding sides from each other:

\[
\frac{\kappa \varphi_a}{r + \kappa \varphi_a} \left( \frac{\lambda_b}{\lambda_a} - 1 \right) r [v(x_2, a) - v(x_0, a)] = -r (x_2 - x_0) \frac{\lambda_b (1 + w(1))}{r + \kappa \varphi_a} \]  

(A.13)

The lhs to this equation is positive because of the monotonicity of \( v(\cdot, a) \), but the rhs is negative, hence a contradiction is resulted, thereby proving the connectedness of \( \mathcal{M}_b \).\]

**Proof of part (iii):**

**High cost regime:** First, I show in this regime \( \mathcal{M}_a \) cannot have a lower boundary point in \( \mathcal{M}_b \), that is \( \partial \mathcal{M}_a \cap \mathcal{M}_b = \emptyset \). Toward the contradiction assume \( \exists y \in \partial \mathcal{M}_a \cap \mathcal{M}_b \). Then, a similar analysis to the previous part yields

\[
\left( \frac{\lambda_b}{\lambda_a} - 1 \right) rv(y, b) = -c \left( \frac{r + \kappa \varphi_b \lambda_b}{\kappa \varphi_b} \lambda_a - 1 \right) + ry \lambda_b (1 + w(1)) \]  

(A.14)

In light of lemma 2, such a \( y \) is a global maximum for \( v(\cdot, b)/v(\cdot, a) \) on the region \( w > 0 \), therefore, conditioned on the existence of the second derivative, it must be non-positive at \( y^+ \), so

\[
\frac{v''(y, b)}{v(y, b)} \leq \frac{v''(y, a)}{v(y, a)} \Rightarrow v''(y, b) \leq \frac{r + \kappa \varphi_b}{\kappa \varphi_b} v''(y, a). \]  

(A.15)

Next, I find an equation for the second derivative by differentiating the HJB equation (2.5) on the continuation region:

\[
rv'(y, q) = \lambda_q (1 + w(1) - v(y, q)) - \lambda_q y v'(y, q)
\]

\[ - \lambda_q (1 - 2y) v'(y, q) - \lambda_q y (1 - y) v''(y, q) \]  

(A.16)
Substituting $v'(\cdot, q)$ from the HJB in the above equation leads to

$$
\lambda_q y (1 - y) v''(y, q) = \lambda_q \left(1 + w(1) - v(y, q)\right) - \frac{(r + \lambda_q (1 - y))}{\lambda_q y (1 - y)} \times \ldots
$$

$$
\ldots \left[ -c + \lambda_q y (1 + w(1)) - (r + \lambda_q y) v(y, q) \right]
= -\frac{r}{1 - y} + \frac{r + \lambda_q}{\lambda_q y (1 - y)} r v(y, q) + \frac{c (r + \lambda_q (1 - y))}{\lambda_q y (1 - y)}.
$$

(A.17)

Plugging the second derivatives from above into (A.15) and applying some rearrangements yield the following equivalent relation

$$
rv(y, b) \left(\frac{\lambda_b}{\lambda_a} - 1\right) \left(1 + \frac{r}{\lambda_a} + \frac{r}{\lambda_b}\right) \geq \left[ry (1 + w(1)) - c(1 - y)\right] \left(\frac{r + \kappa \varphi_b \lambda_b}{\kappa \varphi_b \lambda_a} - 1\right) - \frac{cr}{\lambda_b} \left(\frac{r + \kappa \varphi_b \lambda_b^2}{\kappa \varphi_b \lambda_a^2} - 1\right).
$$

(A.18)

Then, one can substitute (A.14) in above and apply several regroupings to obtain:

$$
y \left\{ (1 + w(1)) \left[\lambda_a (r + \lambda_b) - \kappa \varphi_b (\lambda_b - \lambda_a)\right] - c \left(\lambda_b + r^{-1} \kappa \varphi_b (\lambda_b - \lambda_a)\right) \right\} \geq cr
$$

(A.19)

I would then substitute $w(1)$ from (A.6) in above and get an equivalent conditions to (A.15) that is only in terms of primitives:

$$
\frac{cr^2}{r + \kappa \varphi_b} \left(1 + \frac{\kappa \varphi_b}{r + \lambda_b}\right) + cy \lambda_b \left(1 + \frac{r}{r + \lambda_b \left(\frac{r}{r + \lambda_b + \kappa \varphi_b}\right)}\right) \leq y \left[\lambda_a (r + \lambda_b) - \kappa \varphi_b (\lambda_b - \lambda_a)\right]
$$

(A.20)

Then, I am going to show that the lhs above is always greater than the rhs thus there is no $y \in \partial M_a \cap M_b$. Obviously at $y = 0$ the lhs is greater than the rhs. At $y = 1$, the rhs is increasing in $\lambda_a$, so can be upper-bounded when $\lambda_a$ assumes its maximum level in the high-cost regime, i.e $c + \frac{\kappa \varphi_b (\lambda_b - \lambda)}{r + \lambda_b + \kappa \varphi_b}$. Therefore the rhs of (A.20) at $y = 1$ is upper-bounded as

$$
\lambda_a (r + \lambda_b) - \kappa \varphi_b (\lambda_b - \lambda_a) \leq c (r + \lambda_b).
$$

(A.21)

However, the lhs of (A.20) takes $c(r + \lambda_b)$ at $y = 1$. So (A.20) can never be satisfied, therefore in the high cost regime $M_a$ can not have a lower boundary point in $M_b$. Given $1 \notin M_a$ and the monotonicity of $w$ on $M_a^c$, the only possible candidate for a non-empty $M_a$ is $(\alpha_a, \beta_a)$ such that $\alpha_a < \alpha_b := \inf M_b$. Because of optimality, $v(\cdot, a)$ must smoothly meet the zero
function at $\alpha_a$, so similar analysis to (A.11) would imply $\alpha_a = c/\lambda_a(1 + w(1))$, in that $w(1)$ follows (A.6). Further, the superharmonic condition for $v(\cdot, b)$ requires that at $\pi = \alpha_a$

$$0 \geq [\mathcal{L}_b v](\alpha_a, b) - rv(\alpha_a, b) - c = \lambda_b \alpha_a (1 + w(1)) - c = \left( \frac{\lambda_b}{\lambda_a} - 1 \right). \quad (A.22)$$

However, this never holds, because the rightmost side above is positive. So the only continuation set that survives the high-cost regime is $\mathcal{M}_a = \emptyset$.

**Low cost regime:** Note that in this regime $w(1)$ follows (A.7). I first prove in equilibrium it must be that $\mathcal{M}_a \subset \mathcal{M}_b$. We have seen in the part (i) that $1 \in \mathcal{M}_a \cap \mathcal{M}_b$ in this regime. To show the above set inclusion, I prove $\alpha_a := \inf \mathcal{M}_a \in \mathcal{M}_b$, that is the lowest boundary point of $\mathcal{M}_a$ denoted by $\alpha_a$ is contained in $\mathcal{M}_b$. Toward contradiction assume $\alpha_a < \alpha_b$, where $\alpha_b = \inf \mathcal{M}_b$. Examining the superharmonicity of $v(\cdot, b)$ on $[0, \alpha_a]$ leads to

$$\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c = \lambda_b \pi (1 + w(1)) - c = \frac{\lambda_b}{\lambda_a} \lambda_a \pi (1 + w(1)) - c$$

$$= \frac{\lambda_b}{\lambda_a} \lambda_a (\pi - \alpha_a) (1 + w(1)) + \left( \frac{\lambda_b}{\lambda_a} - 1 \right) c. \quad (A.23)$$

As $\pi$ approaches $\alpha_a$ from below, the first term above converges to zero while the second term remains a positive constant. Therefore, $\exists \pi_0 < \alpha_a$ such that $\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c > 0$ for all $\pi_0 < \pi \leq \alpha_a$. This violates the superharmonicity of $v(\cdot, b)$, so there can be no equilibrium in which the lowest boundary point $\alpha_a \notin \mathcal{M}_b$. Next, I show having $\mathcal{M}_a \subset \mathcal{M}_b$ leads us to the connectedness of $\mathcal{M}_a$. Because of optimality of $v(\cdot, b)$ the principles of continuous and smooth fit hold at $\pi = \alpha_b$ with the zero outside option. Combining this with (2.10) implies the following expression for $v(\cdot, b)$:

$$v(\pi, b) = -\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) \pi$$

$$+ \left\{ \frac{c}{r} - \frac{\lambda_b}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) \alpha_b \right\} \left( \frac{1 - \pi}{1 - \alpha_b} \right)^{1+r/\lambda_b} \left( \frac{\pi}{\alpha_b} \right)^{-r/\lambda_b}, \quad (A.24)$$

with $\alpha_b$ following (A.11). Furthermore, the above value function is convex if and only if

$$\left\{ \frac{c}{r} - \frac{\lambda_b}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) \alpha_b \right\} \geq 0. \quad (A.25)$$
Substituting $\alpha_b$ in this leads to an equivalent condition for convexity:

$$\frac{c}{r} - \frac{c}{r + \lambda} = \frac{c^2}{r(r + \lambda_b)(1 + w(1))} = \frac{c}{r(r + \lambda_b)} \left( \lambda_b - \frac{c}{1 + w(1)} \right) \geq 0. \quad (A.26)$$

The above condition always holds because $\lambda_b > c$ and $w(1) > 0$, therefore $v(\cdot, b)$ followed from (A.24) is a convex function. Now define $[D_a v](\pi, a) := [L_a v](\pi, a) - rv(\pi, a) - c$, and note that from the HJB equation

$$[D_a v](\pi, a) = \frac{-\kappa \varphi_b}{r + \kappa \varphi_b} (\lambda_b - \lambda_a) \frac{rv(\pi, b) + c}{\lambda_b} + \frac{r \lambda_a(1 + w(1)) - cr}{r + \kappa \varphi_b}. \quad (A.27)$$

Consequently, convexity of $v(\cdot, b)$ implies

$$\frac{\partial^2}{\partial \pi^2} [D_a v](\pi, a) = \frac{-\kappa \varphi_b (\lambda_b - \lambda_a)}{(r + \kappa \varphi_b) \lambda_b} v''(\pi, b) < 0. \quad (A.28)$$

Therefore, $[D_a v](\cdot, a)$ is a concave function in $\pi$. Were $\mathcal{M}_a$ not be connected then at least it has two disjoint components, say $(x_1, x_2)$ and $(x_3, x_4)$ where $x_2 < x_3$. Superharmonicity jointly with the satisfaction of Bellman equation on the continuation region require that $[D_a v](\cdot, a)$ is negative just below $x_1$, is zero on $[x_1, x_2]$, becomes negative again on $(x_2, x_3)$, followed by being zero on $(x_3, x_4)$. This pattern is not consistent with the concavity of $[D_a v](\cdot, a)$, therefore $\mathcal{M}_a$ must be connected. \qed

### A.3 Proof of theorem 5

I prove the assertion only for the high-cost regime, as the proof of other case follows the same logic, but is just lengthier. From proposition 4, we know in this regime the only matching sets that survive in the equilibrium are $\mathcal{M}_a = \emptyset$ and $\mathcal{M}_b = (\alpha_b, 1]$, where $\alpha_b$ is found via the continuous and smooth fit principles as

$$\alpha_b = \frac{c}{\lambda_b (1 + w(1))}. \quad (A.29)$$
Also, from the construction of that proposition we know that the following profile embodies the only candidate for an equilibrium with increasing $C^1[0,1]$ value functions on $(\alpha_b, 1]$: 

\begin{align*}
  w(\pi) &= \frac{\kappa \varphi_b}{r + \kappa \varphi_b} v(\pi, b) \\
  v(\pi, a) &= w(\pi) \\
  v(\pi, b) &= -\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) \pi \\
  &\quad + \left\{ \frac{c}{r} - \frac{\lambda_b}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) \alpha_b \right\} \left( \frac{1 - \pi}{1 - \alpha_b} \right)^{1+r/\lambda_b} \left( \frac{\pi}{\alpha_b} \right)^{-r/\lambda_b} \tag{A.30c}
\end{align*}

And all equal to zero on $[0, \alpha_b]$. Therefore, our only task here is to employ a verification scheme to show that the above value functions are indeed the optimal equilibrium values. I divide the proof into three steps: (a) verifying the majorizing and superharmonicity conditions; (b) using these two and applying a Martingale method argument to establish the optimality of the above profile of the value functions; (c) for large $r$ the Banach fixed point theorem is applied and proves the uniqueness of the identified equilibrium in the larger space of bounded continuous functions.

**Step (a):**

Majorizing. This step is quite straightforward because in (A.30) $w = v(\cdot, a)$ and $v(\cdot, b) \geq w = \frac{\kappa \varphi_b}{r + \kappa \varphi_b} v(\cdot, b)$.

Superharmonicity of $v(\cdot, b)$. Obviously the superharmonic condition holds with equality on $(\alpha_b, 1]$ because of the Bellman equation. However, it needs to be checked on $[0, \alpha_b]$ as it carried out below:

\[ [\mathcal{L}_b v](\pi, b) - rv(\pi, b) - c = \lambda_b \pi \left( 1 + w(1) \right) - c \leq \lambda_b \alpha_b \left( 1 + w(1) \right) - c = 0. \tag{A.31} \]

Superharmonicity of $v(\cdot, a)$. Remember that in the high cost regime $\mathcal{M}_a = \emptyset$, thus $v(\cdot, a) = w(\cdot)$. So on $[0, \alpha_b]$:

\[ [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c = \lambda_a \pi \left( 1 + w(1) \right) - c \leq \lambda_a \alpha_b \left( 1 + w(1) \right) - c \leq 0, \tag{A.32} \]

where in the last inequality I used the expression (A.29) for $\alpha_b$, that consequently verifies the superharmonicity on $[0, \alpha_b]$. The analysis of the superharmonicity of $v(\cdot, a)$ on $(\alpha_b, 1]$
however needs a little more work:

\[
[L_a v](\pi,a) - rv(\pi,a) - c = \left[ L_a \left( \frac{\kappa \phi_b}{r + \kappa \phi_b} \right) \right](\pi, b) - \frac{r \kappa \phi_b}{r + \kappa \phi_b} v(\pi, b) - c \\
= -\frac{\kappa \phi_b}{r + \kappa \phi_b} ((L_b v)(\pi,b) - rv(\pi,b) - c) \\
= -\frac{\kappa \phi_b}{r + \kappa \phi_b} (\lambda_b - \lambda_a) \pi (1 + w(1) - v(\pi,b) - (1 - \pi)v'(\pi,b)) \\
+ \frac{r \lambda_a \pi}{r + \kappa \phi_b} (1 + w(1)) - \frac{cr}{r + \kappa \phi_b}
\]

Some straightforward manipulations analogous to equation (A.25) implies the candidate \(v(\cdot, b)\) in (A.30) is also convex, therefore, \(v(\pi,b) + (1 - \pi)v'(\pi,b) \leq v(1,b)\) that yields an upper bound on the above relation:

\[
[L_a v](\pi,a) - rv(\pi,a) - c \leq \left( -\frac{\kappa \phi_b}{r + \kappa \phi_b} \left( \lambda_b - \lambda_a \right) \pi \left( 1 + w(1) + \frac{c}{r} \right) + \frac{r \lambda_a \pi (1 + w(1))}{r + \kappa \phi_b} - \frac{cr}{r + \kappa \phi_b} \right)^+ \\
\leq \left( -\frac{\kappa \phi_b}{r + \kappa \phi_b} \frac{r \left( \lambda_b - \lambda_a \right) \pi}{r + \lambda_b} \left( 1 + w(1) + \frac{c}{r} \right) + \frac{r \lambda_a \left( 1 + w(1) \right)}{r + \kappa \phi_b} - \frac{cr}{r + \kappa \phi_b} \right)^+
\]

(A.34)

In the second inequality above I used the fact that the rhs of the first inequality is negative at \(\pi = 0\). Now denote the argument of \((\cdot)^+\) by \(3\). It is increasing in \(\lambda_a\), hence can be bounded above when \(\lambda_a\) is replaced with \(c + rw(1)\) (its maximum value in the high-cost regime):

\[
3 \leq -\frac{\kappa \phi_b}{r + \kappa \phi_b} \frac{r \left( \lambda_b - c - rw(1) \right) \left( 1 + w(1) + \frac{c}{r} \right)}{r + \lambda_b} + \frac{r \left( c + rw(1) \right) \left( 1 + w(1) \right) - cr}{r + \kappa \phi_b} \\
= -\frac{\kappa \phi_b}{r + \kappa \phi_b} \frac{r (\lambda_b - c) (r + \lambda_b) (r + \kappa \phi_b + c)}{r (\kappa \phi_b + r + \lambda_b)^2} + \frac{\kappa \phi_b}{r + \kappa \phi_b} \frac{(\lambda_b - c) (r + \lambda_b) (r + \kappa \phi_b + c)}{r (\kappa \phi_b + r + \lambda_b)^2} = 0,
\]

(A.35)

where in the second line \(w(1)\) is replaced from (A.6). This concludes the superharmonicity of \(v(\cdot, a)\) w.r.t \(L_a\) on \((\alpha, 1]\), and hence on the entire unit interval.

**Step (b):** Define \(v(\iota, \pi, q) := v(\pi, q)1_{(\iota = 0)} + (\iota + w(\pi))1_{(\iota = 1)}\), where \(\iota\) is the success indicator.
process. Since \( v \) is a bounded function, for each \( q \in \{a, b\} \), one can find a bounded (and hence uniformly integrable) Martingale process \( M^q \) such that:

\[
e^{-rt}v(t, \pi_t, q) = v(t, \pi, q) + \int_0^t e^{-rs} [L_q v(\cdot, \cdot, q) - rv(\cdot, \cdot, q)] (t_{s^-}, \pi_{s^-}) ds + M^q_t \tag{A.36}
\]

In that \( L_q v(t, \pi, q) = (L_q v(\pi, q)) 1_{t=0} \). From the majorant condition, for every stopping time \( \tau \), we have \( v(t_{\tau}, \pi_{\tau}, q) \geq t_{\tau} + w(\pi_{\tau}) \), therefore

\[
e^{-r\tau} (t_{\tau} + w(\pi_{\tau})) \leq v(t, \pi, q) + \int_0^{\tau} e^{-rs} [L_q v(\cdot, \cdot, q) - rv(\cdot, \cdot, q)] (t_{s^-}, \pi_{s^-}) ds + M^q_{\tau} \tag{A.37}
\]

where in the second inequality I used the superharmonic property proven before. Applying Doob’s optional stopping theorem yields \( E M^q_{\tau} = 0 \), hence for every stopping time \( \tau \):

\[
v(t, \pi, q) \geq E_{\pi, q, t} \left[ e^{-r\tau} (t_{\tau} + w(\pi_{\tau})) - c \int_0^{\tau} e^{-rs} ds \right] \tag{A.38}
\]

That in turn implies

\[
v(\pi, q) \geq \sup_{\tau} E_{\pi, q, t=0} \left[ e^{-r\tau} (t_{\tau} + w(\pi_{\tau})) - c \int_0^{\tau} e^{-rs} ds \right]. \tag{A.39}
\]

Now for each \( q \), let \( \tau(q) := \inf \{ t \geq 0 : \pi_t \notin M_q \text{ or } t = 1 \} \) that is the optimal stopping policy. Using this in (A.36) yields

\[
e^{-r\tau(q)} (t_{\tau(q)} + w(\pi_{\tau(q)})) = e^{-r\tau(q)} v(t_{\tau(q)}, \pi_{\tau(q)}, q)
\]

\[
= v(t, \pi, q) + \int_0^{\tau(q)} e^{-rs} [L_q v(\cdot, \cdot, q) - rv(\cdot, \cdot, q)] (t_{s^-}, \pi_{s^-}) ds + M^q_{\tau(q)}
\]

\[
= v(t, \pi, q) - \int_0^{\tau(q)} ce^{-rs} ds + M^q_{\tau(q)}, \tag{A.40}
\]

which after taking expectations of both sides amounts to

\[
v(t, \pi, q) = E_{\pi, q, t} \left[ e^{-r\tau(q)} (t_{\tau(q)} + w(\pi_{\tau(q)})) - c \int_0^{\tau(q)} e^{-rs} ds \right], \tag{A.41}
\]

therefore concluding the verification proof and the theorem.
Step (c): I slightly change the notation only in this part and denote $v_q(\cdot) := v(\cdot, q)$. Then, for every $(v_a, v_b, w) \in C[0, 1]$, define

$$T_q w(\pi) := \sup_{\tau} \left\{ E_q \left[ e^{-r\sigma} - c \int_0^\sigma e^{-rs}ds + e^{-r\sigma} w(\pi_\sigma); \sigma \leq \tau \right] \\
+ E_q \left[ -c \int_0^\tau e^{-rs}ds + e^{-r\tau} w(\pi_\tau); \sigma > \tau \right] \right\} \text{ for } q \in \{a, b\}, \quad (A.42a)$$

$$T_0[v_a, v_b, w](\pi) := r^{-1} \kappa \sum_{q \in M(\pi)} [v_q(\pi) - w(\pi)] \varphi_q, \quad (A.42b)$$

where $E_q$ is the expectation w.r.t to the Poisson process with intensity $\lambda_q$ and $M(\pi) = \{q : v_q(\pi) > w(\pi)\}$. Define $T := (T_a, T_b, T_0)$. The goal of this part of the proof is to show the fixed-point of $T$ exists and is unique. Given the definition of $M(\pi)$ one can see that $T_0$ preserves the continuity. Then, I show $T_q (C[0, 1]) \subset C[0, 1]$. For this assume $w \in C[0, 1]$ and rewrite (A.42a) as

$$T_q w(\pi) = \sup_{\tau} \{ E_q [Z_1(\sigma); \sigma \leq \tau] + E_q [Z_2(\tau); \sigma > \tau] \}. \quad (A.43)$$

For the first term,

$$E_q [Z_1(\sigma); \sigma \leq \tau] = \int_0^\infty Z_1(t)P_q (t < \sigma \leq t + dt, \sigma \leq \tau)$$

$$= \int_0^\infty Z_1(t)P_q (t < \sigma \leq t + dt) P_q (\sigma \geq t | t < \sigma \leq t + dt) \quad (A.44)$$

$$= \int_0^\tau Z_1(t) \lambda_q \pi e^{-\lambda_q t} dt.$$

And for the second term,

$$E_q [Z_2(\tau); \sigma > \tau] = E_q [E_q [Z_2(\tau)1_{\{\sigma > \tau\}} | I_\tau]] = E_q [Z_2(\tau)P_q (\sigma > \tau | I_\tau)]$$

$$= Z_2(\tau) \pi e^{-\lambda_q \tau}, \quad (A.45)$$

where in last line, I used the fact that $\tau$ is inevitably $I_0$-measurable because of the Poissonian underlying process, and $\pi_\tau$ is the posterior belief at $\tau$ given that the success will not have
arrived by then. Hence,

\[ T_q w(\pi) = \sup_{\tau} \left\{ \pi \int_0^\tau Z_1(t)\lambda_q e^{-\lambda_q t} dt + Z_2(\tau)\pi e^{-\lambda_q \tau} \right\}. \] (A.46)

Because of the Bayes law, \( \frac{\pi_{\tau \pi}}{1-\pi} = \frac{\phi}{1-\phi} e^{-\lambda_q \tau} \), so \( \pi_{\tau} \) is continuous in the initial belief \( \pi \). Therefore, the above representation together with the continuity of \( w \) amount to the continuity of \( T_0 w \). So, we can now deduce that \( T: (C[0, 1])^3 \to (C[0, 1])^3 \).

The next step is to investigate the contraction property of \( T \). For this, let us equip \((C[0, 1])^3\) with the following norm,

\[ \|(v_a, v_b, w)\|_c := \varsigma \left( \|v_a\|_\infty + \|v_b\|_\infty + \|w\|_\infty \right), \] (A.47)

where \( \varsigma > 0 \) is to be determined. First, I examine the contraction coefficient of \( T_q \). For every \( w, \tilde{w} \in C[0, 1] \):

\[ |T_q[w] - T_q[\tilde{w}]| (\pi) \leq \sup_{\tau} \left\{ E_q \left[ e^{-\tau \sigma} |w(\pi_{\sigma}) - \tilde{w}(\pi_{\sigma})| ; \sigma \leq \tau \right] + E_q \left[ e^{-\tau \sigma} |w(\pi_{\tau}) - \tilde{w}(\pi_{\tau})| ; \sigma > \tau \right] \right\} \] (A.48)

\[ \leq \|w - \tilde{w}\|_\infty \sup_{\tau} E_q \left[ e^{-\tau (\sigma \wedge \tau)} \right] = \|w - \tilde{w}\|_\infty. \]

Let \( \phi := \varphi_a + \varphi_b \) be the total steady state mass of startups, and let \( v, \tilde{v} \in (C[0, 1])^2 \), that respectively enforce the matchings sets \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \). Then:

\[ (T_0[v, w] - T_0[\tilde{v}, \tilde{w}]) (\pi) = r^{-1} \kappa \left( \sum_{q \in \mathcal{M}(\pi)} (v_q(\pi) - w(\pi)) \varphi_q - \sum_{q \in \tilde{\mathcal{M}}(\pi)} (\tilde{v}_q(\pi) - \tilde{w}(\pi)) \varphi_q \right) \\
= r^{-1} \kappa \sum_{q \in \mathcal{M}(\pi) \setminus \tilde{\mathcal{M}}(\pi)} (v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)) \varphi_q \\
+ r^{-1} \kappa \sum_{q \in \mathcal{M}(\pi) \cap \tilde{\mathcal{M}}(\pi)} (v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)) \varphi_q \\
+ r^{-1} \kappa \sum_{q \in \tilde{\mathcal{M}}(\pi) \setminus \mathcal{M}(\pi)} (\tilde{v}_q(\pi) - \tilde{w}(\pi)) \varphi_q - r^{-1} \kappa \sum_{q \in \tilde{\mathcal{M}}(\pi) \setminus \mathcal{M}(\pi)} (\tilde{v}_q(\pi) - \tilde{w}(\pi)) \varphi_q \\
\leq r^{-1} \kappa \phi \left( \sum_{q \in (a, b)} \|v_q - \tilde{v}_q\|_\infty + \|w - \tilde{w}\|_\infty \right). \] (A.49)
Putting together the preceding bounds yields:

$$\| T[(v_a, v_b, w)] - T[\tilde{(v_a, v_b, w)}] \|_c = \varsigma \left( \| T_a[w] - T_a[\tilde{w}] \|_{\infty} + \| T_b[w] - T_b[\tilde{w}] \|_{\infty} \right)$$

$$+ \| T_0[(v_a, v_b, w)] - T_0[(\tilde{v}_a, \tilde{v}_b, \tilde{w})] \|_{\infty}$$

$$\leq 2\varsigma \| w - \tilde{w} \|_{\infty} + r^{-1} \kappa \phi \left( \sum_{q \in \{a,b\}} \| v_q - \tilde{v}_q \|_{\infty} + \| w - \tilde{w} \|_{\infty} \right)$$

$$= r^{-1} \kappa \phi \| v_a - \tilde{v}_a \|_{\infty} + r^{-1} \kappa \phi \| v_b - \tilde{v}_b \|_{\infty}$$

$$+ \left( 2\varsigma + r^{-1} \kappa \phi \right) \| w - \tilde{w} \|_{\infty}$$

(A.50)

Assume $r^{-1} \kappa \phi < 1/3$, and find $\varepsilon > 0$ such that $r^{-1} \kappa \phi < 1/(1 + \varepsilon)(3 + 2\varepsilon)$, and let $\varsigma = (1 + \varepsilon)r^{-1} \kappa \phi$, then

$$\| T[(v_a, v_b, w)] - T[\tilde{(v_a, v_b, w)}] \|_c \leq \frac{r^{-1} \kappa \phi}{\varsigma} \times$$

$$\left( \varsigma \| v_a - \tilde{v}_a \|_{\infty} + \varsigma \| v_b - \tilde{v}_b \|_{\infty} + \varsigma \left( 2\varsigma + r^{-1} \kappa \phi \right) \| w - \tilde{w} \|_{\infty} \right)$$

$$\leq \frac{1}{1 + \varepsilon} \| (v_a, v_b, w) - (\tilde{v}_a, \tilde{v}_b, \tilde{w}) \|_c.$$ 

(A.51)

So the contraction is resulted, and the Banach fixed-point theorem implies that there exists a unique fixed-point in the space of bounded continuous functions, so long as $r > 3\kappa \phi$. 

**A.4 Proof of proposition 6**

Let us emphasize that the planner’s problem is still subject to search frictions and incomplete information about the VCs’ types. That is the hypothetical planner only knows that a fraction $p$ of VCs have high types. In this regard, the only choice variable would be the matching sets $M_a$ and $M_b$. Specifically, the venture capitalists can not decide whether to match or not upon being contacted by the entrepreneurs. Further, they have no control right on when to terminate the funding. In the planner’s problem all these rights are conferred to the benevolent planner. So, equation (2.8) carries through but with the indicator functions
chosen by the planner. The planner’s problem thus reduces to maximizing

\[ S(\mathcal{M}) = n(1)w(1) + m_a(1)v(1, a) + m_b(1)v(1, b), \]  

in that \( n(1) := G(\{1\}) \), \( m_a(1) := F(\{1\}, \{a\}) \), \( m_b(1) := F(\{1\}, \{b\}) \) and \( n(\alpha) := G(\{\alpha\}) \) subject to (3.2). The solution to this system when \( \chi_a(1) = \chi_b(1) = 1 \) is

\[ (n(\alpha), n(1), m_a(1), m_b(1)) = \left( \frac{1 - p \ (p - \alpha)/(1 - \alpha)}{1 - \alpha}, \frac{\kappa \varphi_a (p - \alpha)/(1 - \alpha)}{1 + \frac{\kappa \varphi_a + \kappa \varphi_b}{\lambda_a}}, \frac{\kappa \varphi_b (p - \alpha)/(1 - \alpha)}{1 + \frac{\kappa \varphi_a + \kappa \varphi_b}{\lambda_b}} \right), \]

(A.53)

when \( \chi_a(1) = 1 \) and \( \chi_b(1) = 0 \) is

\[ (n(\alpha), n(1), m_a(1)) = \left( \frac{1 - p \ (p - \alpha)/(1 - \alpha)}{1 - \alpha}, \frac{(p - \alpha)/(1 - \alpha)}{1 + \frac{\lambda_a}{\kappa \varphi_a}} \right), \]

(A.54)

and finally when \( \chi_a(1) = 0 \) and \( \chi_b(1) = 1 \):

\[ (n(\alpha), n(1), m_b(1)) = \left( \frac{1 - p \ (p - \alpha)/(1 - \alpha)}{1 - \alpha}, \frac{(p - \alpha)/(1 - \alpha)}{1 + \frac{\lambda_b}{\kappa \varphi_b}} \right). \]

(A.55)

Denote the social welfare function in the first case by \( S_{a,b} \), in the second case by \( S_a \) and lastly in the third case by \( S_b \), then

\[ S_{a,b} = \frac{(p - \alpha)/(1 - \alpha)}{1 + \frac{\kappa \varphi_a + \kappa \varphi_b}{\lambda_a}} \left( \frac{\kappa \varphi_a v(1, a) + \kappa \varphi_b v(1, b)}{r + \kappa \varphi_a + \kappa \varphi_b} + \frac{\kappa \varphi_a}{\lambda_a} v(1, a) + \frac{\kappa \varphi_b}{\lambda_b} v(1, b) \right), \]

(A.56a)

\[ S_a = \frac{(p - \alpha)/(1 - \alpha)}{1 + \kappa \varphi_a/\lambda_a} \left( \frac{\kappa \varphi_a v(1, a)}{r + \kappa \varphi_a} + \frac{\kappa \varphi_a}{\lambda_a} v(1, a) \right), \]

(A.56b)

\[ S_b = \frac{(p - \alpha)/(1 - \alpha)}{1 + \kappa \varphi_b/\lambda_b} \left( \frac{\kappa \varphi_b v(1, b)}{r + \kappa \varphi_b} + \frac{\kappa \varphi_b}{\lambda_b} v(1, b) \right). \]

(A.56c)

Conditioned on \( \chi_q(1) = 1 \), we have \((r + \lambda_q)v(1, q) = -c + \lambda_q(1 + w(1))\). Using this and the planner’s version of (2.8), one can apply some algebraic simplifications on the above
expressions and obtain:

\[
\begin{align*}
  rS_a &= \frac{p - \alpha - \frac{\kappa \varphi_a}{\lambda_a}}{1 - \alpha + \frac{\kappa \varphi_a}{\lambda_a}} \left( \lambda_a - c \right) \quad (A.57a) \\
  rS_b &= \frac{p - \alpha - \frac{\kappa \varphi_b}{\lambda_b}}{1 - \alpha + \frac{\kappa \varphi_b}{\lambda_b}} \left( \lambda_b - c \right) \quad (A.57b) \\
  rS_{a,b} &= \frac{p - \alpha}{1 - \alpha + \frac{\kappa \varphi_a}{\lambda_a} + \frac{\kappa \varphi_b}{\lambda_b}} \left( \frac{\kappa \varphi_a}{\lambda_a} \left( \lambda_a - c \right) + \frac{\kappa \varphi_b}{\lambda_b} \left( \lambda_b - c \right) \right) \quad (A.57c)
\end{align*}
\]

Suppose \( \alpha \) is the lowest boundary point of \( \mathcal{M}_q \), then continuity requires \( v(\alpha, q) = 0 \), which is also reinforced by the principle of optimality. For otherwise, if \( v(\alpha, q) > 0 \) one can reduce \( \alpha \) and increase the welfare functions while leaving all value functions still positive. In a neighborhood above \( \alpha \) the value function \( v(\cdot, q) \) takes the form of

\[
v(\pi, q; \alpha) = \frac{c}{r} + \frac{\lambda_q}{r + \lambda_q} \left( 1 + w(1) + \frac{c}{r} \right) \pi + \left\{ \frac{c}{r} - \frac{\lambda_q}{r + \lambda_q} \left( 1 + w(1) + \frac{c}{r} \right) \alpha_q \right\} \left( \frac{1 - \pi}{1 - \alpha_q} \right)^{1 + r/\lambda_q} \left( \frac{\pi}{\alpha_q} \right)^{-r/\lambda_q}.
\]

At \( \pi = \alpha \),

\[
\frac{\partial v}{\partial \pi} \bigg|_{\pi=\alpha} = \frac{(1 + w(1)) \alpha \lambda_q - c}{\lambda_q (1 - \alpha) \alpha},
\]

so a necessary condition for \( v \) to be increasing is \( \alpha \geq \frac{c}{\lambda_q (1 + w(1))} \). The welfare expressions in (A.57) are decreasing in \( \alpha \), compatible with the learning effect. So choosing \( \alpha \) exactly equal to \( \frac{c}{\lambda_q (1 + w(1))} \) leads to an upper-bound on the welfare function, that could be attained if one finds its corresponding matching sets. This argument provides another support for the smooth-fit principle at the lowest boundary point. In addition, on \( \pi \geq \alpha \),

\[
\frac{\partial^2 v}{\partial \pi \partial \alpha} = \frac{(r + \lambda_q \pi) [(1 + w(1)) \alpha \lambda_q - c]}{\lambda_q^2 \alpha^2 (1 - \alpha)^2} \left( \frac{1 - \pi}{1 - \alpha} \right)^{r/\lambda_q} \left( \frac{\pi}{\lambda_q} \right)^{-(1+r/\lambda_q)} \geq 0, \quad (A.60)
\]

which implies that \( \partial v / \partial \pi \) remains positive for all \( \pi \geq \alpha \) and in \( \mathcal{M}_q \). As claimed before the lowest boundary point is in the set \{ \( c / \lambda_q (1 + w(1)) : q = a, b \) \}. So long as \( \lambda_b > \lambda_a \) the efficient choice is to set \( \alpha \) as the lowest boundary point of \( \mathcal{M}_b \), and hence \( \alpha = \frac{c}{\lambda_q (1 + w(1))} \).

One can now verify that in the high-cost regime \( S_b \) is the largest of all in (A.57), and in the low-cost regime \( S_{a,b} \) is the largest. Therefore, the equilibrium matching sets found in proposition 4 are in fact constrained efficient.
A.5 Social optimum in section 5

The planner maximizes the present value of social surplus $S$ that is presented in equation (5.4), subject to the population dynamics in (5.5) and (5.6). The instruments that the planner has at her disposal is the choice of matching sets: $\{\chi_q(\pi) : q \in \{a, b\} \text{ and } \pi \in [0, 1]\}$. The current value Hamiltonian for this problem is

$$
H = \sum_q \left( (\lambda_q - c) m_q(1) + \int (\lambda_q \pi - c) m_q(\pi) d\pi \right)^{
+ \sum_q v_q(1, q) \left[ -\lambda_q m_q(1) + \kappa \varphi_q n(1) \chi_q(1) \right]^{
+ \sum_q \int v_q(\pi, q) \left[ -\lambda_q \pi m_q(\pi) + \kappa \varphi_q n(\pi) \chi_q(\pi) + \lambda_q \partial_\pi (\pi (1 - \pi) m_q(\pi)) \right] d\pi
+ \int w_q(\pi) \left[ -\sum_q \kappa \varphi_q n(\pi) \chi_q(\pi) \right] d\pi
+ \rho \left[ \zeta \lambda_a \left( m_a(1) + \int \pi m_a(\pi) d\pi \right) - \kappa \lambda_b \left( n(1) \chi_a(1) + \int n(\pi) \chi_q(\pi) d\pi \right) \right].
$$

(A.61)

Applying the integration by part implies that

$$
\int v_q(\pi, q) \lambda_q \partial_\pi (\pi (1 - \pi) m_q(\pi)) d\pi = - \int \lambda_q \pi (1 - \pi) v_q'(\pi, q) m_q(\pi) d\pi.
$$

(A.62)

Substituting this in the Hamiltonian and regrouping with respect to the population measures amount to

$$
H = \sum_q m_q(1) \left[ \lambda_q - c + \lambda_q \left( w_q(1) - v_q(1, q) \right) + \rho \zeta \lambda_a 1_{\{q=a\}} \right]^{
+ n(1) \left[ \sum_q \kappa \varphi_q (v_q(1, q) - w_q(1)) \chi_q(1) - \rho \kappa \varphi_b \chi_b(1) \right]^{
+ \sum_q \int m_q(\pi) \left[ \lambda_q \pi - c + \lambda_q \pi (w_q(1) - v_q(\pi, q)) - \lambda_q \pi (1 - \pi) v_q'(\pi, q) + \rho \zeta \lambda_a \pi 1_{\{q=a\}} \right] d\pi
+ \int n(\pi) \left[ \sum_q \kappa \varphi_q (v_q(\pi, q) - w_q(\pi)) \chi_q(\pi) - \rho \kappa \varphi_b \chi_b(\pi) \right] d\pi.
$$

(A.63)
The planner’s optimization problem, as expressed above, features a *continuum* of control and state processes. Therefore, I appeal to the heuristic method of Van Imhoff and Ritzen (1988) (chapter 6) to interpret the integrals as the summation of discrete variables over intervals of length $d\pi$. The first implication of the above representation is that from the planner’s viewpoint the optimal matching indicator $\chi^*$ satisfies:

$$\chi^*_q(\pi) = 1 \iff v^*_q(\pi, q) > w^*_q(\pi),$$  \hspace{1cm} (A.64)

that is a match is socially optimal if the social marginal value of the partnership (i.e $v^*$) dominates the social marginal value of reputation while being unmatched (i.e $w^*$).

Next, I express the co-state equations for each of the social marginal values. In that, I will use the Gâteaux derivative (see chapter 7 in Aliprantis and Border (2006)) of the Hamiltonian w.r.t the associated probability measure. For instance, to find out the derivative of $H$ w.r.t $n(x)$ (for $x < 1$), define $\delta_x$ as the Dirac mass concentrated at $x$,

$$D_{n(x)}H := \lim_{\epsilon \to 0} \frac{H[n(x) + \epsilon \delta_x] - H[n(x)]}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \epsilon \delta_x(\pi) \left[ \sum_q \kappa \varphi_q (v^*_q(\pi, q) - w^*_q(\pi)) \chi^*_q(\pi) - \rho \kappa \varphi_b \chi^*_b(\pi) \right] d\pi$$  \hspace{1cm} (A.65)

Hence the co-state equations are ordered as follows:

$$rv^*_q(1, q) - \dot{v}^*_q(\pi, q) = D_{m_q(1)}H = \lambda_q - c + \lambda_q (w^*_q(1) - v^*_q(1, q)) + \rho \zeta \lambda a 1_{q=a}$$

$$rw^*_q(1) - \dot{w}^*_q(1) = D_{n(1)}H = \sum_q \kappa \varphi_q [v^*_q(1, q) - w^*_q(1)] \chi^*_q(1) - \rho \kappa \varphi_b \chi^*_b(1)$$

$$rv^*_q(\pi, q) - \dot{v}^*_q(\pi, q) = D_{m_q(\pi)}H$$

$$= \lambda_q \pi - c + \lambda_q \pi (w^*_q(1) - v^*_q(\pi, q)) - \lambda_q \pi (1 - \pi)v'_q(\pi, q) + \rho \zeta \lambda a \pi 1_{q=a}$$

$$rw^*_q(\pi) - \dot{w}^*_q(\pi) = D_{n(\pi)}H = \sum_q \kappa \varphi_q [v^*_q(\pi, q) - w^*_q(\pi)] \chi^*_q(\pi) - \rho \kappa \varphi_b \chi^*_b(\pi)$$  \hspace{1cm} (A.66)

The social shadow value of the mass of late stage projects, i.e $\rho$, satisfies the following
first-order condition:

\[ r \rho - \dot{\rho} = \frac{\partial H}{\partial \varphi_b} \]

\[ = \kappa n(1) (v_s(1, b) - w_s(1) - \rho) \chi_b^*(1) + \int \kappa n(\pi) (v_s(\pi, b) - w_s(\pi) - \rho) \chi_b^*(\pi) d\pi \]

(A.67)

In the steady state the above representation leads to (5.12).

A.6 Proof of proposition 8

I need the following lemma to prove the proposition.

Lemma 13. In any normal equilibrium \( w(\cdot) \) is increasing iff \( \{v(\cdot, a), v(\cdot, b)\} \) are increasing.

Proof. In the normal equilibria \( w(\pi) \) follows

\[ w(\pi) = \max_{\chi} \left\{ \frac{r^{-1}\kappa \psi(\pi) [v(\pi, a) \varphi_a \chi_a(\pi) + v(\pi, b) \varphi_b \chi_b(\pi)]}{\mu + r^{-1}\kappa \psi(\pi) [\varphi_a \chi_a(\pi) + \varphi_b \chi_b(\pi)]} \right\} \]

(A.68)

Assume first, that \( \{v(\cdot, a), v(\cdot, b)\} \) are increasing. It is known that the maximum of increasing functions remains increasing, therefore I have to show for any combination of \( \chi \)'s the \( \text{rhs} \) of the above expression is increasing in \( \pi \). For example, let \( \chi_a = \chi_b = 1 \), then its derivative is positively proportional to

\[ r^{-1}\kappa \psi(\pi) (v'(\pi, a) \varphi_a + v'(\pi, b) \varphi_b) + r^{-1}\kappa \mu \psi'(\pi) (v(\pi, a) \varphi_a + v(\pi, b) \varphi_b) \geq 0. \]

(A.69)

The other permutations of \( \chi_a, \chi_b \) can also be checked, and one can similarly verify that for each combination, the \( \text{rhs} \) is increasing in \( \pi \), therefore \( w(\cdot) \) becomes increasing.

Conversely, now assume \( w(\cdot) \) is increasing. Then the same analysis presented in lemma 3 implies that \( \{v(\cdot, a), v(\cdot, b)\} \) become increasing. ||

Proof of part (i): For proving the existence of a normal equilibrium, I first establish the existence of a fixed-point \( \alpha_e \) to the system (6.3) and (6.4). To fix ideas, let us define the
following mappings $M : [0, 1] \to [0, 1]$, $W : [0, 1] \to \mathbb{R}_+$ and $A : \mathbb{R}_+ \to [0, 1]$:  

$$M(x) := \frac{1 - p}{1 - x} \psi(x) + \frac{p - x}{1 - x}$$  

$$W(\mu) := \max_x \left\{ \frac{r^{-1} \kappa \left[ \varphi_b (\lambda_b - c) (r + \lambda_a) \chi_b (1) + \varphi_a (\lambda_a - c) (r + \lambda_b) \chi_a (1) \right]}{(r + \lambda_a) (r + \lambda_b) \mu + \kappa \varphi_b (r + \lambda_a) \chi_b (1) + \kappa \varphi_a (r + \lambda_b) \chi_a (1)} \right\}$$  

$$A(w) := \frac{c}{\lambda_b (1 + w)}$$  

(A.70)  

Then, $\alpha_e$ is the fixed point of $\mathfrak{F} : [0, 1] \to [0, 1]$, where $\mathfrak{F} := A \circ W \circ M$. Since this map is continuous on $[0, 1]$, the existence of fixed-point is obvious. However, the normal equilibrium requires $\alpha_e < p$. For this note that $\mathfrak{F}(0) > 0$ and $\mathfrak{F}(p) = c / \lambda_b (1 + W(\psi(p))) < c / \lambda_b < p$.  

(A.71)  

The mean-value theorem therefore implies that there always exists a normal equilibrium with $0 < \alpha_e < p$.  

Now I analyze the best-response correspondence for a generic VC. Suppose all VCs except one follow the investment strategy induced by $M_b = (\alpha_e, 1]$ and $M_a \subset M_b$. Then, $\mu = M(\alpha_e)$. Using the machinery developed in proposition 4 and the previous lemma, one can easily confirm the unique best-response of the potential deviant VC is the above matching sets $(M_a, M_b)$. ||  

**Proof of part (ii):** Assuming $\psi'' \leq 0$ implies that  

$$M'(x) = \frac{1 - p}{1 - x} \left( \psi'(x) - \frac{1 - \psi(x)}{1 - x} \right) \geq 0.$$  

(A.72)  

Hence, the composition map becomes increasing from $[0, 1]$ to itself, because $W$ and $A$ are both decreasing. Further, define  

$$\Psi := \{ \psi : [0, 1] \to [0, 1] | \psi(0) = 0, \psi(1) = 1, \psi' \geq 0, \psi'' \leq 0 \};$$  

(A.73)  

and endow $\Psi$ with the pointwise order $\succeq$, i.e $\psi_2 \succeq \psi_1$ iff $\psi_2(x) \geq \psi_1(x)$, $\forall x \in [0, 1]$. So, $(\Psi, \succeq)$ becomes a partially ordered set that is used as the underlying parameter space for the fixed-point map $\mathfrak{F}$. With slight abuse of notation, I extend the domain of $\mathfrak{F}$ as $\mathfrak{F} : [0, 1] \times \Psi \to [0, 1]$. Holding $x$ constant, $\mathfrak{F}(x, \psi)$ is increasing in $\psi$ w.r.t $\succeq$ order. Therefore, the mapping $\mathfrak{F}$ is an increasing function from $[0, 1] \times \Psi$ to $[0, 1]$. Now one can apply corollaries 2.5.1 and 2.5.2 of *Topkis (1998)* to conclude that the set of fixed-points is a complete lattice.
and its greatest (least) element is increasing in $\psi \in \Psi$. Finally, the lattice of fixed-points, i.e $\alpha_e$'s, completely Pareto rank the equilibria. Because smaller values of $\alpha_e$ lead to smaller $\mu$ and hence larger $w(1)$ and \{v(1,b), v(1,a)\}. In addition, it is associated to larger masses of \{n(1), m(1,b), m(1,a)\}. Therefore, the welfare ranking of equilibria coincides inversely with the ranking of fixed-points of $\mathfrak{g}$.

\[ \Box \]

A.7 Proof of lemma 9

In the steady state the time derivatives in (6.7) must be equal to zero, therefore (6.7a) and (6.7b) amount to:

\[ n(1) = \frac{\delta + \lambda}{\kappa \varphi / \mu} m(1) \]  
\[ (\delta + \lambda + \kappa \varphi / \mu) \delta m(1) = \kappa \varphi / \mu \int_\alpha^p \lambda \pi m(\pi) d\pi. \]  

(A.74a)  
(A.74b)

Also at $\pi = p$, equation (6.7c) implies that $n(p) = \delta / (\delta + \kappa \varphi \psi(p)/\mu)$. Next, the expression found in (6.9) translates to

\[ \int_\alpha^p \pi m(\pi) d\pi = \frac{\lambda m(\alpha)}{\delta + \lambda} T_2(\alpha). \]  

(A.75)

The rhs to (6.10) can be simplified using the steady state ODE resulted from $\dot{m}(\pi) = 0$:

\[ \kappa \varphi \frac{\psi(p)}{\mu} n(p) = \frac{\delta \kappa \varphi \psi(p)/\mu}{\delta + \kappa \varphi \psi(p)/\mu} = \lambda \int_\alpha^p \pi m(\pi) d\pi + \delta \int_\alpha^p m(\pi) d\pi + \delta n(\alpha) \]
\[ = \lambda \int_\alpha^p \partial_\pi (\pi(1-\pi)m(\pi)) d\pi + \delta n(\alpha) \]
\[ = \lambda [p(1-p)m(p) - \alpha(1-\alpha)m(\alpha)] + \delta n(\alpha) \]
\[ = \lambda m(\alpha) Y_1(\alpha) + \delta n(\alpha) \]

(A.76)

Recall that because of Bayesian learning over matches the steady state average reputation must be equal to $p$:

\[ m(1) + n(1) + pn(p) + \int_\alpha^p \pi m(\pi) d\pi + an(\alpha) = p \]  

(A.77)
Simplifying this relation using (A.74b) and (A.75) implies

$$\lambda m(\alpha) \Upsilon_2(\alpha) + \alpha \delta n(\alpha) = \frac{\delta \kappa \varphi(p) / \mu}{\delta + \varphi(p) / \mu} \mathbf{p}.$$  \hspace{1cm} (A.78)

It is now straightforward to solve for \(n(\alpha), m(\alpha)\) using (A.76) and (A.78), thereby obtaining (6.11d) and

$$m(\alpha) = \frac{\delta \kappa \varphi(p) / \mu}{\lambda (\delta + \varphi(p) / \mu)} \mathbf{p} - \alpha \Upsilon_2(\alpha) - \alpha \Upsilon_1(\alpha).$$  \hspace{1cm} (A.79)

Substituting \(m(\alpha)\) from above into (A.75) yields the lemma’s claim for \(\int_\alpha^\mathbf{p} \pi m(\pi) \, d\pi\), i.e. equation (6.11b). Subsequently, \(m(1)\) can be found from (A.74b) thus verifying (6.11c). Finally, from the second line in (A.76) one obtains the following expression

$$\int_\alpha^\mathbf{p} m(\pi) \, d\pi = \frac{\lambda m(\alpha)}{\delta} \left( \Upsilon_1(\alpha) - \frac{\lambda}{\delta + \lambda} \Upsilon_2(\alpha) \right),$$  \hspace{1cm} (A.80)

that amounts to (6.11a) by substituting \(m(\alpha)\) in the above expression.

A.8 Stochastic ordering results in subsection 6.2

For a better understanding of the stochastic ordering on the steady distribution of \(\pi_\infty\), I would first express the CDF of the density \(m(\cdot)\):

$$\int_\alpha^\pi m(x) \, dx = \frac{\kappa \varphi(p) / \mu}{\delta + \varphi(p) / \mu} \mathbf{p} - \alpha \Upsilon_2(\alpha, \mathbf{p}) - \alpha \Upsilon_1(\alpha, \mathbf{p}) \left[ \frac{\delta}{\delta + \lambda} \Upsilon_{2,1}(\alpha, \pi) + \Upsilon_{1,2}(\alpha, \pi) \right],$$

$$\Upsilon_{i,j}(x, y) := \left( \frac{y}{x} \right)^{(\delta/\lambda+1)} \left( \frac{1-y}{1-x} \right)^{-(\delta/\lambda+2)} y^i (1-y)^j - x^i (1-x)^j$$  \hspace{1cm} (A.81)

In addition, using the solution found for \(m(\pi)\) and the expression (A.79) for \(m(\alpha)\) it is easy to verify that for \(\pi \in [\alpha, \mathbf{p}]\)

$$m(\pi) = \frac{\delta \kappa \varphi(p) / \mu}{\lambda (\delta + \varphi(p) / \mu)} \left( \frac{\pi}{\mathbf{p}} \right)^{(\delta/\lambda-1)} \left( \frac{1-\pi}{1-\mathbf{p}} \right)^{-(\delta/\lambda+2)} \frac{1}{\mathbf{p}(1-\mathbf{p})};$$  \hspace{1cm} (A.82)

therefore for a fixed \(\mu\) the above density is independent of \(\alpha\).

My next goal is to show that \(M(\mu, \alpha)\) is increasing in each argument holding the other one constant. For this, I appeal to the theory of stochastic orders, and in particular I employ the second-order stochastic dominance. For two real-valued random variables \(X\) and \(Y\), it
is said that $X \gtrless_{\text{SSD}} Y$ if $Eu(X) \geq Eu(Y)$ for every increasing and concave function $u$. An equivalent definition is that $X \gtrless_{\text{SSD}} Y$ if $E[(X - t)_-] \geq E[(Y - t)_-]$ for every $t \in \mathbb{R}$ provided that the expectations exist. The next lemma offers a sufficient condition for second-order stochastic dominance that originates from the work of Karlin et al. (1963).

**Lemma 14** (Sufficient condition for SSD). *Suppose the following two conditions hold:*

(i) $E[X] \geq E[Y].$

(ii) There exists $t_0 \in \mathbb{R}$ such that for all $t \leq t_0$, $P(X \geq t) \geq P(Y \geq t)$ and for all $t > t_0$, $P(X \geq t) \leq P(Y \geq t)$.

*Then* $X \gtrless_{\text{SSD}} Y$.

**Proof.** For every $t \leq t_0$,

$$
E[(X - t)_-] = -\int_0^\infty P(-(X - t) > u) \, du = -\int_0^\infty P(X < t - u) \, du
$$

$$
= -\int_{-\infty}^t P(X < z) \, dz \geq -\int_{-\infty}^t P(Y < z) \, dz = E[(Y - t)_-].
$$

(A.83)

Also, an equivalent representation for $E[(X - t)_-]$ is

$$
E[(X - t)_+] = E[(X - t); X < t]
$$

$$
= E[X] - t - E[X - t; X \geq t] = E[X] - t - \int_t^\infty P(X \geq z) \, dz.
$$

(A.84)

Therefore,

$$
E[(X - t)_-] - E[(Y - t)_-] = E[X] - E[Y] + \int_t^\infty [P(Y \geq z) - P(X \geq z)] \, dz.
$$

(A.85)

The first term is positive and the integral term is also positive for all $t > t_0$, so $E[(X - t)_-] \geq E[(Y - t)_-]$ for $t > t_0$ as well.

I will use the technique offered in this lemma to prove that an increase in $\alpha$ or $\mu$ positively shifts the steady state distribution of $\pi_\infty$. This distribution is completely described by the measures found in lemma 9. For every Borel subset $B \subset [0, 1]:$

$$
P(\pi_\infty \in B) = (m(1) + n(1)) \delta_1(B) + n(p)\delta_p(B) + \int_B m(\pi) \, d\pi + n(\alpha)\delta_\alpha(B)
$$

(A.86)

For every $r \in \mathbb{R}$, $(r)_- := \min\{r, 0\}$. The reader can refer to chapter 4 of Shaked and Shanthikumar (2007) for the proof of the equivalence.
Lemma 15. Let $\alpha_1 \leq \alpha_2 < p$ and $\mu_1 \leq \mu_2$, then

(i) Holding $\alpha$ constant, $\pi_\infty(\mu_2) \succeq_{SSD} \pi_\infty(\mu_1)$.

(ii) Holding $\mu$ constant, $\pi_\infty(\alpha_2) \succeq_{SSD} \pi_\infty(\alpha_1)$.

Proof. Part (i): I show that

$$
P(\pi_\infty(\mu_2) \geq t) = \begin{cases} P(\pi_\infty(\mu_1) \geq t) & \forall t \leq p \\ \leq P(\pi_\infty(\mu_1) \geq t) & \forall t > p. \end{cases}
$$

(A.87)

Note that for every $t > p$

$$
P(\pi_\infty \geq t) = m(1) + n(1) = \frac{\kappa \varphi \psi(p)/\mu \lambda}{\delta + \kappa \varphi \psi(p)/\mu \lambda} \frac{(p - \alpha) \Upsilon_{2,1}(\alpha, p) - \alpha \Upsilon_{1,1}(\alpha, p)}{\Upsilon_{2,1}(\alpha, p) - \alpha \Upsilon_{1,1}(\alpha, p)}
$$

(A.88)

that is obviously decreasing in $\mu$, hence proving the second assertion in (A.87). For every $t \leq p$,

$$
P(\pi_\infty \geq t) = 1 - P(\pi_\infty < t) = 1 - \left( n(\alpha) + \int_\alpha^t m(\pi)d\pi \right)
$$

(A.89)

According to (6.11d), the mass $n(\alpha)$ is decreasing in $\mu$, so is $\int_\alpha^t m(\pi)d\pi$ according to (A.81). Hence, $P(\pi_\infty \geq t)$ must be increasing in $\mu$ for every $t \leq p$, thus establishing the first line of (A.87). Given that $E[\pi_\infty(\mu_2)] = E[\pi_\infty(\mu_1)] = p$, then both parts of lemma 14 are satisfied to conclude part (i).

Part (ii): Holding $\mu$ constant, for every $t \leq \alpha_2$

$$
P(\pi_\infty(\alpha_2) \geq t) = 1 \geq P(\pi_\infty(\alpha_1) \geq t).
$$

(A.90)

Alternatively, for every $t > \alpha_2$

$$
P(\pi_\infty(\alpha) \geq t) = 1_{t \leq p} \left( \int_t^p m(\pi)d\pi + n(p) \right) + n(1) + m(1)
$$

(A.91)

Because of (A.82) the integral term is independent of $\alpha$ (for a fixed $\mu$). This is the case for $n(p)$ as well. Therefore, it is sufficient to show holding $\mu$ constant, $n(1) + m(1)$ is decreasing
in $\alpha$. This is equivalent to verifying the following expression is decreasing in $\alpha$:

$$\frac{(p - \alpha)\Upsilon_{2,1}(\alpha, p)}{\Upsilon_{2,1}(\alpha, p) - \alpha \Upsilon_{1,1}(\alpha, p)} = \frac{(p - \alpha)\Upsilon_{2,1}(\alpha, p)}{(p - \alpha)p(1 - p)} \left(\frac{p}{\alpha}\right)^{\delta/\lambda - 1} \left(\frac{1 - p}{1 - \alpha}\right)^{-(\delta/\lambda + 2)}$$

$$= p \left[1 - \frac{\alpha^2(1 - \alpha)}{p^2(1 - p)} \left(\frac{\alpha}{p}\right)^{\delta/\lambda - 1} \left(\frac{1 - \alpha}{1 - p}\right)^{-(\delta/\lambda + 2)}\right]$$

$$= p \left[1 - \left(\frac{\alpha}{1 - \alpha}\right)^{\delta/\lambda + 1} \left(\frac{p}{1 - p}\right)^{-(\delta/\lambda + 1)}\right]$$

(A.92)

Since $\alpha/(1 - \alpha)$ is increasing in $\alpha$, then the above expression is decreasing in $\alpha$, so as a result of this, for every $\alpha_1 < \alpha_2 < p$ and $t > \alpha_2$:

$$P(\pi_\infty(\alpha_2) \geq t) \leq P(\pi_\infty(\alpha_1) \geq t)$$

(A.93)

Hence, lemma 14 can be applied to conclude part (ii). \hfill \Box

## B General type space

The goal of this appendix is to extend the results of section 2 to the general type space for projects. Specifically, I show there always exists an increasing reputation function $w$ that satisfies the investors fixed-point problem. Suppose the startups’ types are drawn from an arbitrary distribution with CDF $\phi(\cdot)$ and a bounded support $[a, b]$. The success arrival intensity takes the general form of $\lambda_q(\theta)$, for which I denote $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \lambda_q$, and assume $\lambda_q \leq \bar{\lambda}_q \leq \lambda$ for all $q \in \text{Supp}(\phi)$.

The reputation value function satisfies

$$w(\pi) = \frac{\kappa}{r} \int [v(\pi, q) - w(\pi)]^+ \phi(dq),$$

therefore for every subset $B \subset [a, b]$, one can see the equilibrium value functions $(w, v)$ satisfy

$$w(\pi) \geq \frac{\kappa}{r} \int_B [v(\pi, q) - w(\pi)] \phi(dq) \Rightarrow w(\pi) \geq \frac{\int_B v(\pi, q)\phi(dq)}{1 + \frac{\kappa}{r} \phi(B)}.$$  

(B.2)

Setting $B^* = \{q : v(\pi, q) > w(\pi)\}$ to bind the above inequality, one can propose the following equivalent representation for the reputation value function:

$$w(\pi) = \sup \left\{ \frac{\int_B v(\pi, q)\phi(dq)}{1 + \frac{\kappa}{r} \phi(B)} : B \subset [a, b]\right\}$$

(B.3)
On the other hand, given the reputation function \( w \), each investor solves the stopping time problem when matched with a project of type \( q \):

\[
v(\pi, q) = \sup_{\tau} \mathbb{E} \left[ 1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_{0}^{\sigma \wedge \tau} e^{-rs} ds + e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau}) \right]
\]  

(B.4)

For a given \( q \), let \( T_q w \) be the matching value function resulted from the above stopping time problem, hence from (B.3) it follows that \( w \) is the fixed-point to the following operator:

\[
A w := \sup \left\{ \int B T_q w \phi(dq) \middle| B \subset [a, b] \right\}
\]

(B.5)

In what follows I propose the appropriate function space on which \( A \) will be defined, and advance the study of its fixed-point with its properties.

Let \( L^1[0, 1] \) be the Banach space of Lebesgue integrable functions on the unit interval, and \( L^1_+[0, 1] \) be the subset of nonnegative functions which is readily seen to be a cone. Let \( \succcurlyeq \) be the partial order induced by the cone \( L^1_+[0, 1] \) on the Banach space \( L^1[0, 1] \), that is \( w_2 \succcurlyeq w_1 \) if \( w_2(\pi) \geq w_1(\pi), \forall \pi \in [0, 1] \). Then, it readily follows from (B.4) that \( T_q \) is a positive and monotone operator, that is letting \( 0 \) to be the zero element of \( L^1[0, 1] \), then \( T_q 0 \succcurlyeq 0 \), and \( T_q w_2 \succcurlyeq T_q w_1 \) for \( w_2 \succcurlyeq w_1 \) in \( L^1_+[0, 1] \). Further, it can easily be verified that \( A \) inherits positivity and monotonicity from the collection \( \{ T_q : q \in [a, b] \} \). Next, I show without loss of generality, we can restrict the search for the fixed-point to the bounded region of all \( w \in L^1_+[0, 1] \) where \( \|w\|_\infty \leq \lambda/r \).

**Lemma 16.** For every \( w \in L^1_+[0, 1] \),

\[
\|T_q w(\cdot)\| \leq \max \left\{ \|w\|, \frac{\lambda}{r + \lambda} (1 + \|w\|) \right\}, \quad \phi \text{ almost surely.}
\]  

(B.6)

**Proof.** For every \( q \in \text{Supp}(\phi) \),

\[
T_q w(\pi) = \sup_{\tau} \mathbb{E} \left[ 1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_{0}^{\sigma \wedge \tau} e^{-rs} ds + e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau}) \right]
\]

\[
\leq \sup_{\tau} \mathbb{E} \left[ 1_{\{\sigma \leq \tau\}} e^{-r\sigma} + e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau}) \right]
\]

\[
\leq \sup_{\tau} \mathbb{E} \left[ \max \left\{ e^{-r\tau} w(\pi_{\tau}), e^{-r\sigma} (1 + w(\pi_{\sigma})) \right\} \right]
\]

\[
\leq \max \left\{ \|w\|, \mathbb{E} \left[ e^{-r\sigma} \right] (1 + \|w\|) \right\}.
\]

(B.7)

A cone is a subset \( K \) of a Banach space which is (i) closed, (ii) for every \( x, y \in K \) and \( \alpha, \beta \geq 0 \): \( \alpha x + \beta y \in K \), and (iii) \( K \cap (-K) = 0 \). Henceforth, if not stated explicitly all norms are the sup-norm.
For a fixed \( \pi \in [0, 1] \) and \( q \in [a, b] \)

\[
E[e^{-r\sigma}] = \pi \int_0^\infty e^{-rt} \bar{\lambda}_q e^{-\lambda_q t} dt + (1 - \pi) \int_0^\infty e^{-rt} \lambda_q e^{-\lambda_q t} dt
= \pi \frac{\lambda_q}{r + \lambda_q} + (1 - \pi) \frac{\lambda_q}{r + \Delta_q} \leq \frac{\lambda}{r + \lambda}
\]  

(B.8)

Substituting this into the upper bound found above for \( T_q w(\pi) \) concludes the proof.

I use the previous lemma to limit the search for the space of fixed-points.

**Lemma 17.** Any fixed-point of \( A \) (if exists) is order bounded above by the constant function \( \lambda / r \).

**Proof.** First, note that the supremum in (B.5) is achieved by \( B_w = \{ q : T_q w(\pi, q) > w(\pi) \} \) for any candidate fixed-point \( w \). Then, for any such candidate

\[
\left( 1 + \frac{\kappa}{r} \phi(B_w) \right) w(\pi) = \int_{B_w} T_q w(\pi) \phi(dq),
\]

(B.9)

therefore, using the result of the previous lemma

\[
\left( 1 + \frac{\kappa}{r} \phi(B_w) \right) \|w\| \leq \max \left\{ \|w\|, \frac{\lambda}{r + \lambda} (1 + \|w\|) \right\} \phi(B_w).
\]

(B.10)

Assume to the contrary that \( \|w\| > \lambda / r \), then \( \max \left\{ \|w\|, \frac{\lambda}{r + \lambda} (1 + \|w\|) \right\} = \|w\| \), and (B.10) amounts to

\[
\left( 1 + \frac{\kappa}{r} \phi(B_w) \right) \|w\| \leq \|w\| \phi(B_w).
\]

(B.11)

Cancelling \( \|w\| \) from both sides implies \( 1 + \frac{\kappa}{r} \phi(B_w) \leq \phi(B_w) \). Since it was assumed \( \|w\| > \lambda / r \), then \( \phi(B_w) > 0 \). On the other hand \( \phi(B_w) \leq 1 \). These two together with (B.11) yield the contradiction and hence the proof of the lemma.

**Definition 18** (Regular and strongly-minihedral cones: Krasnoselskij (1964) sections 1.5 and 1.7). A Banach space partially ordered by means of a cone is called *regularly partially ordered*, if any monotone-increasing sequence, *order-bounded* from above, converges in norm to a limit point. A cone which generates a regular partial ordering is called a regular cone. A cone is said to be *strongly minihedral* if every order bounded subset has a least upper bound (order supremum).

Now consider the Banach space of integrable functions \( L^1[0, 1] \), and the positive cone of \( L^1_+[0, 1] = \{ f \in L^1[0, 1] : f(x) \geq 0 \ \forall x \in [0, 1] \} \). This cone is regular, and for any monotone
increasing sequence \( \{f_n\} \subset L^1[0, 1] \) such that \( f_1 \lesssim f_2 \lesssim \ldots \) and order bounded from above, \( \|f_n - f\|_{L^1} \to 0 \) where \( f(x) = \sup_n f_n(x) \) for every \( x \in [0, 1] \) (Dominated convergence theorem). In addition \( L^1_+[0, 1] \) is strongly minihedral (page 52 Krasnoselskij (1964)).

Let \( (0, \lambda/r) := \{ f \in L^1_+[0, 1] : 0 \lesssim f \lesssim \lambda/r \} \) be the order interval of nonnegative \( L^1 \) functions, order bounded above by the constant function \( \lambda/r \). In light of the lemma 16, we have \( T_q : (0, \lambda/r) \to (0, \lambda/r) \) for every \( q \in [a, b] \) and hence \( A : (0, \lambda/r) \to (0, \lambda/r) \).

At this stage, I can apply part (a) of theorem 4.1 in Krasnoselskij (1964) to conclude the existence of a fixed-point of \( A \) in \( (0, \lambda/r) \), because the mapping \( A \) is monotonic in a strongly minihedral cone space. However, the mere existence of the fixed-point is far from enough. In particular, we want to know whether there exists a continuous and/or increasing fixed-point for \( A \). To answer such questions, I will need to dig deeper into the mapping \( A \), beyond its monotonicity. In doing so, I shall construct a monotone sequence of functions, and show it converges in the \( L^1 \) sense to a fixed-point of \( A \).

Fix \( w_0 := 0 \) and recursively define \( w_n = A w_{n-1} \), therefore \( \{w_n\} \subset (0, \lambda/r) \) is an increasing sequence order bounded from above, hence converges in \( L^1 \) to \( w_\infty \in (0, \lambda/r) \) where \( w_\infty(\pi) = \sup_n w_n(\pi) \) for each \( \pi \in [0, 1] \) (because of the regularity of the \( L^1_+[0, 1] \) cone). The conceptual merit of this recursive construction is summarized in the following two points:

(i) Say a property \( * \) is owned by \( w_0 \), and is preserved by the mapping \( A \). Then, it holds along the sequence \( \{w_n\} \).

(ii) If \( * \) is stable under the \( L^1 \) limit, then \( w_\infty \) holds this property.

Therefore, if \( A \) is \( L^1 \) continuous along the sequence \( \{w_n\} \), then \( w_\infty \) becomes the fixed-point and the presumptive property \( * \) will be inherited to the fixed-point.

**Proposition 19.** For the sequence \( \{w_n\} \) defined above, it holds that \( \|A w_n - A w_\infty\|_{L^1} \to 0 \), and as a result \( w_\infty = A w_\infty \).

**Proof.** First note that for every \( \pi \in [0, 1] \),

\[
A w_\infty(\pi) - A w_n(\pi) = \sup \left\{ \frac{\int_B T_q w_\infty(\pi) \phi(dq)}{1 + \frac{\pi}{r} \phi(B)} : B \subset [a, b] \right\} - \sup \left\{ \frac{\int_B T_q w_n(\pi) \phi(dq)}{1 + \frac{\pi}{r} \phi(B)} : B \subset [a, b] \right\} \\
\leq \sup \left\{ \frac{\int_B (T_q w_\infty - T_q w_n)(\pi) \phi(dq)}{1 + \frac{\pi}{r} \phi(B)} : B \subset [a, b] \right\} \\
\leq \int_0^1 (T_q w_\infty - T_q w_n)(\pi) \phi(dq),
\]

(B.12)
where in the last line I used the fact that $w_\infty \gtrless w_n$ and the monotonicity of the operator $T_q$. Therefore, the $L^1$-norm can be bounded above as:

$$\|A w_\infty - A w_n\|_{L^1} = \int_0^1 (A w_\infty(\pi) - A w_n(\pi)) \, d\pi \leq \int_0^1 \int_0^1 (T_q w_\infty - T_q w_n)(\pi) \phi(dq) \, d\pi = \int_0^1 \|T_q w_\infty - T_q w_n\|_{L^1} \phi(dq)$$

(B.13)

For the last equality relation, I used the fact that the integrand is positive and uniformly bounded above by $\lambda/r$ to apply the Fubini’s theorem and exchange the order of integrations. Since the integrand of the last integral above is uniformly bounded (over all $q \in [a, b]$), then one can use the Lebesgue-dominated-convergence theorem to get:

$$\lim_{n \to \infty} \|A w_\infty - A w_n\|_{L^1} \leq \lim_{n \to \infty} \int_0^1 \|T_q w_\infty - T_q w_n\|_{L^1} \phi(dq) = \int_0^1 \lim_{n \to \infty} \|T_q w_\infty - T_q w_n\|_{L^1} \phi(dq)$$

(B.14)

Next, I propose a method to upper-bound $(T_q w_\infty - T_q w_n)(\pi)$, and hence its $L^1$-norm. For this let $G$ represent the random variable inside the expectation operator in the definition of $T_q w$:

$$(T_q w_\infty - T_q w_n)(\pi) = \sup_\tau E_\pi[G(\sigma, w_\infty; \tau)] - \sup_\tau E[G(\sigma, w_n; \tau)] \leq \sup_\tau E_\pi[e^{-r(\sigma \wedge \tau)}(w_\infty - w_n)(\pi_{\sigma \wedge \tau})]$$

$$\leq E_\pi[e^{-r\sigma}(w_\infty - w_n)(\pi_\sigma)] + \sup_\tau E_\pi[e^{-r\tau}(w_\infty - w_n)(\pi_\tau); \tau < \sigma]$$

(B.15)

Therefore, the $L^1$-norm is bounded by

$$\|T_q w_\infty - T_q w_n\|_{L^1} \leq \int_0^1 E_\pi\left[e^{-r\sigma}(w_\infty - w_n)(\pi_\sigma)\right] \, d\pi + \int_0^1 \sup_\tau E_\pi\left[e^{-r\tau}(w_\infty - w_n)(\pi_\tau); \tau < \sigma\right] \, d\pi$$

(B.16)

The integrands of both integrals are bounded by $\lambda/r$, hence applying the Lebesgue-dominated-
convergence theorem twice for the first integral implies

$$
\lim_{n \to \infty} \mathcal{I}_1 = \int_0^1 \lim_{n \to \infty} E_\pi \left[ e^{-r\sigma} (w_\infty - w_n) (\pi_\sigma) \right] d\pi = \int_0^1 E_\pi \left[ \lim_{n \to \infty} e^{-r\sigma} (w_\infty - w_n) (\pi_\sigma) \right] d\pi = 0,
$$

(B.17)

because $w_\infty$ is the pointwise supremum of the sequence $\{w_n\}$. To show the convergence for the second integral, first note that for every given $\varepsilon > 0$ one can find $T > 0$ such that

$$
\sup_{\tau} E_\pi \left[ e^{-r\tau} (w_\infty - w_n)(\pi_\tau); \tau < \sigma \right] \leq \sup_{\tau \leq T} E_\pi \left[ e^{-r\tau} (w_\infty - w_n)(\pi_\tau); \tau < \sigma \right] + \varepsilon,
$$

(B.18)

uniformly over all $\pi$. This is indeed due to the uniform boundedness of $(w_\infty - w_n)$ by $\lambda/r$. Next, because of the property of supremum for every $\varepsilon > 0$, there exist $\tau_{n,\pi}$ (possibly depending on $n$ and $\pi$) such that

$$
\sup_{\tau \leq T} E_\pi \left[ e^{-r\tau} (w_\infty - w_n)(\pi_\tau); \tau < \sigma \right] \leq e^{-r\tau_{n,\pi}} (w_\infty - w_n)(\pi_{\tau_{n,\pi}}) P_\pi (\tau_{n,\pi} < \sigma) + \varepsilon.
$$

(B.19)

Therefore,

$$
\mathcal{I}_2 \leq \int_0^1 e^{-r\tau_{n,\pi}} (w_\infty - w_n)(\pi_{\tau_{n,\pi}}) P_\pi (\tau_{n,\pi} < \sigma) d\pi + 2\varepsilon
$$

(B.20)

$$
= \int_0^1 e^{-r\tau_{n,\pi}} (w_\infty - w_n)(\pi_{\tau_{n,\pi}}) \left( \pi e^{-\lambda q \tau_{n,\pi}} + (1 - \pi) e^{-\Delta q \tau_{n,\pi}} \right) d\pi + 2\varepsilon.
$$

Because of the Bayes-law, $\pi_{\tau_{n,\pi}} = \frac{\pi e^{-\lambda q \tau_{n,\pi}}}{1 - \pi + \pi e^{-\Delta q \tau_{n,\pi}}}$. Leveraging this relation and applying the change of variable to the above integral lead to

$$
\mathcal{I}_2 - 2\varepsilon \leq \int_0^1 (w_\infty - w_n)(x) \frac{e^{(\lambda_q - 2\Delta_q - r)\tau_{n,x}}}{(1 - x + xe^{\Delta q \tau_{n,x}})^3} dx
$$

(B.21)

$$
\leq \int_0^1 (w_\infty - w_n)(x) e^{(\lambda_q - 2\Delta_q - r)\tau_{n,x}} dx,
$$

where in the last inequality I used the fact that $(1 - x + xe^{\Delta q \tau_{n,x}})$ is increasing in $x$. Since, $\tau_{n,x} \leq T$ the last integrand in (B.21) is uniformly bounded for all $x$ and $n$. Hence, one can apply the Lebesgue-dominated-convergence theorem and obtain

$$
\lim_{n \to \infty} \mathcal{I}_2 \leq \int_0^1 \lim_{n \to \infty} (w_\infty - w_n)(x) e^{(\lambda_q - 2\Delta_q - r)\tau_{n,x}} dx + 2\varepsilon = 2\varepsilon.
$$

(B.22)
Since this relation holds for every $\varepsilon > 0$, then $\lim_{n \to \infty} I_2 = 0$. This establishes the $L^1$ convergence of $A w_n$ to $A w_\infty$ and thus proves that $w_\infty = A w_\infty$.

A very important property owned by $w_0$ and preserved under $A$ is being increasing in $\pi$. In the next lemma, using the techniques from coupling of probability measures and stochastic dominance, I show $A w$ is increasing in $\pi$ when $w$ is.

**Lemma 20.** Let $w$ be an increasing function in $\pi$, then $A w$ becomes increasing in $\pi$ as well.

**Proof.** Fix $q$ and suppose $\pi_2 \geq \pi_1$. Define the random variables

$$\sigma_i \overset{d}{=} \pi_i \exp(\lambda_q) + (1 - \pi_i) \exp(\Lambda_q), \quad i \in \{1, 2\}$$

(B.23)

as the exponential time of success arrivals under $\pi_1$ and $\pi_2$. One can easily check $\sigma_1 \succeq \sigma_2$ in the sense of first order stochastic dominance (see the supplementary material). Therefore, for every decreasing function $f$ we will have $E[f(\sigma_2)] \geq E[f(\sigma_1)]$. Recall the definition of $T_q$:

$$T_q w(\pi) = \sup_{\tau} E_\pi [G(\sigma; \tau)]$$

(B.24)

$$G(\sigma; \tau) := 1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_0^{\sigma \wedge \tau} e^{-rs} ds + e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau}).$$

The first two terms in $G$ are clearly decreasing in $\sigma$, so for every $q \in [a, b]$ and $\tau$:

$$E \left[ 1_{\{\sigma_2 \leq \tau\}} e^{-r\sigma_2} - c \int_0^{\sigma_2 \wedge \tau} e^{-rs} ds \right] \geq E \left[ 1_{\{\sigma_1 \leq \tau\}} e^{-r\sigma_1} - c \int_0^{\sigma_1 \wedge \tau} e^{-rs} ds \right]$$

(B.25)

The proof for monotonicity of the last term in $G$ is a bit more tricky, because $\pi_{\sigma \wedge \tau}$ is not just a function of $\sigma$, but it also depends on the initial $\pi$. So let us define $w(\pi, \sigma; \tau) := e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau})$ where $\pi$ is the initial belief value. To proceed, I need to define $\sigma_1$ and $\sigma_2$ on the same probability space, because the analysis to be presented needs more than the application of the first order stochastic dominance. For this, I use the Strassen theorem (Lindvall (2002) chapter 4) to find the coupling $(\hat{\sigma}_1, \hat{\sigma}_2)$ such that $\hat{\sigma}_i \overset{d}{=} \sigma_i$ for $i = 1, 2$, and crucially $\hat{\sigma}_1 \geq \hat{\sigma}_2$ almost surely. It is proven in the online appendix that for every $\tau$, $w$ is

\[\text{The term } \exp(\lambda) \text{ denotes an exponential random variable with the rate } \lambda.\]
increasing in \( \pi \) and decreasing in \( \sigma \) (while holding \( \pi \) constant), therefore

\[
E_{\pi_2} \left[ e^{-r(\sigma_2 \wedge \tau)} w(\pi_2, \sigma_2 \wedge \tau) \right] = E \left[ w(\pi_2, \hat{\sigma}_2; \tau) \right] \\
\geq E \left[ w(\pi_1, \hat{\sigma}_2; \tau) \right] \\
\geq E \left[ w(\pi_1, \hat{\sigma}_1; \tau) \right] \\
= E_{\pi_1} \left[ e^{-r(\sigma_1 \wedge \tau)} w(\pi_1, \sigma_1 \wedge \tau) \right].
\]

(B.26a)

(B.26b)

(B.26c)

(B.26d)

In (B.26a) and (B.26d), I used the fact that coupling preserves the marginal distributions. In (B.26b), I apply the increasing property of \( w \) in \( \pi \), and in (B.26c) its decreasing property in \( \sigma \).

Combining (B.25) and (B.26) implies that for every \( \tau \) and \( q \in [a, b] \):

\[
E_{\pi_2} [G(\sigma_2; \tau)] \geq E_{\pi_1} [G(\sigma_1; \tau)],
\]

therefore, applying the supremum on both sides (w.r.t to \( \tau \)) yields \( \mathbb{T}_q w(\pi_2) \geq \mathbb{T}_q w(\pi_1) \). From this and expression (B.5), it is now straightforward to conclude that \( A w(\pi_2) \geq A w(\pi_1) \).

Now we are in a position to claim the existence of a fixed-point that is increasing, the proof of which follows from previous lemma and the fact that increasing property is closed under the \( L^1 \) limit.

**Theorem 21.** The operator \( A \) has an increasing fixed-point function.

For a candidate increasing fixed-point \( w \), we can now assure that if \( w(\pi') > 0 \) for some \( \pi' \), then \( w(\pi'') > 0 \) for all \( \pi'' > \pi' \). This means once \( w \) exceeds zero it will never fall down to zero again, therefore the union of all matching sets over \( q \in [a, b] \) must be an increasing set in \([0, 1]\), hence there exists an equilibrium point \( \alpha \) such that

\[
\bigcup_{q \in [a, b]} \{ \pi : \mathbb{T}_q w(\pi) > w(\pi) \} = (\alpha, 1].
\]

(B.27)

Next, I show how \( \alpha \) is determined. Its location is important because it represents the point of endogenous exit from the market. In particular, the VCs with lower reputation than \( \alpha \) would no longer invest. In the next proposition, I show under some natural assumptions, \( \alpha \) is the boundary point of the stopping time problem that a generic investor solves when is matched to the best type of projects, i.e \( q = b \). For this I present two notions. The profile of arrival intensity \( \lambda = \{(\Delta_q, \tilde{\lambda}_q) : q \in [a, b]\} \) is called monotone if \( \Delta_q \) and \( \tilde{\lambda}_q \) are increasing in \( q \). It satisfies the increasing-differences if \( \tilde{\lambda}_{q''} - \Delta_{q''} \geq \tilde{\lambda}_{q'} - \Delta_{q'} \) for every \( q'' > q' \) in \([a, b]\).
Proposition 22. Assume the profile $\lambda$ is monotone and satisfies the increasing-differences. Then, $\alpha$ is the lowest boundary point of $M_b$, and is the unique fixed-point of

$$\alpha = \frac{c}{\Delta_b \left(1 + w \left(\frac{\lambda_b\alpha}{\Delta_{\alpha} + \Delta_b}\right)\right)} - \frac{\lambda_b}{\Delta_b}. \quad (B.28)$$

Proof. Assume by contradiction that $\alpha \notin \text{cl}(M_b)$, and there exists $q < b$ such that $\alpha = \inf M_q$, that is a VC matched with a project of type $q$, terminates the funding as her reputation nears $\alpha$. The principles of optimality requires smooth and continuous fit at $\alpha$, namely $v'(\alpha, q) = v(\alpha, q) = 0$. From the Bellman equation for every $\pi \in M_q$ it must be that

$$rv(\pi, q) = -c + \left(\bar{\lambda}_q\pi + \lambda_q(1 - \pi)\right) \left(1 + w \circ j(\pi) - v(\pi, q)\right) - \pi(1 - \pi)\Delta_q v'(\pi, q). \quad (B.29)$$

In that $j$ returns the posterior after the success has taken place at time $t$:

$$j(\pi_{t^-}) := \frac{\lambda_q\pi_{t^-}}{\lambda_q\pi_{t^-} + \lambda_q(1 - \pi_{t^-})} \quad (B.30)$$

In the baseline model, the success event was conclusive thus $j(\pi) = 1$ for every $\pi \in (0, 1]$. The optimality principles at $\pi = \alpha$ imply

$$c = \left(\alpha\Delta_q + \lambda_q\right) \left(1 + w \circ j(\alpha)\right). \quad (B.31)$$

Further, at $\pi = \alpha$, since $\alpha \notin \text{cl}(M_b)$ then $v(\alpha, b) = w(\alpha) = 0$ and superhamonicity implies that

$$0 > L_b v(\alpha, b) - rv(\alpha, b) - c = (\alpha\Delta_b + \lambda_b)(1 + w \circ j(\alpha)) - c. \quad (B.32)$$

Replacing (B.31) in the above inequality and canceling $c$ from both sides amount to

$$0 > \frac{\alpha\Delta_b + \lambda_b}{\alpha\Delta_q + \lambda_q} - 1. \quad (B.33)$$

However the rhs of the above inequality is positive because of the monotonicity and increasing-differences, hence the contradiction is resulted. Therefore, it must be that $\alpha = \inf M_b$.

On the uniqueness of $\alpha$, note that the lhs of (B.28) is increasing in $\alpha$, while the rhs is decreasing – because $w$ is an increasing function. Therefore, upon the existence, $\alpha$ is
uniquely determined by this equation.

References


C Supplementary proofs

C.1 Proof of lemma 2

Assume equation (2.7) holds, then one can check with $\chi_a(\pi) = \chi_b(\pi) = 1$ in equation (2.8) both of the conditions $v(\pi, a) > w(\pi)$ and $v(\pi, b) > w(\pi)$ are satisfied, therefore the if part is established. For the only if direction, assume $\pi \in \mathcal{M}(a) \cap \mathcal{M}(b)$, then it must be that
\( \chi_a(\pi) = \chi_b(\pi) = 1 \). Replacing this in (2.8) and simplifying \( v(\pi, b) > w(\pi) \) results in the first inequality in (2.7). Similarly, simplifying \( v(\pi, a) > w(\pi) \) leads the second inequality in (2.7).

C.2 Proofs of section B

Proof for \( \sigma_1 \gtrless_{\text{FSD}} \sigma_2 \). For the two random variables defined in (B.23) we have

\[
P(\sigma_i > t) = \pi_i e^{-\tilde{\lambda}_q t} + (1 - \pi_i) e^{-\lambda_q t} \tag{C.1}
\]

therefore,

\[
P(\sigma_1 > t) - P(\sigma_2 > t) = (\pi_2 - \pi_1) \left( e^{-\Delta_q t} - e^{-\tilde{\lambda}_q t} \right) \geq 0, \tag{C.2}
\]

because \( \tilde{\lambda}_q \geq \Delta_q \) for every \( q \in [a, b] \). Therefore, \( \sigma_1 \gtrless_{\text{FSD}} \sigma_2 \).

Properties of the transformed function \( w \). Here I prove the properties claimed about the function \( w \), namely the fact that it is increasing in \( \pi \) (initial belief) and decreasing in \( \sigma \) (success arrival time).

Decreasing in \( \sigma \). Fix the initial belief \( \pi \) (as well as \( \tau \) and \( q \)), then \( w \) is clearly continuous in \( \sigma \) and is constant on \( [\tau, \infty) \). Further, it is decreasing on \( [0, \tau] \), because \( \tilde{\lambda}_q \geq \Delta_q \) so the posterior belief about \( \{\theta = H\} \) falls more as the elapsed time to success gets longer. Formally, because of Bayesian learning

\[
\pi_{\sigma} = \pi_{\sigma^-} + \Delta \pi_{\sigma} \\
= \pi_{\sigma^-} + \frac{\tilde{\lambda}_q - \Delta_q}{\pi_{\sigma^-}(\tilde{\lambda}_q - \Delta_q) + \Delta_q}(1 - \pi_{\sigma^-}), \tag{C.3}
\]

where the first term \( \pi_{\sigma^-} \) is the posterior belief just before the success arrival and the second term \( \Delta \pi_{\sigma} \) is the amount that the posterior jumps up at the time of the success. Define \( \Delta_q := \tilde{\lambda}_q - \Delta_q \geq 0 \), then again because of the Bayes-law:

\[
\pi_{\sigma^-} = \frac{\pi e^{-\Delta_q \sigma}}{1 - \pi + \pi e^{-\Delta_q \sigma}} \Rightarrow \frac{d\pi_{\sigma^-} \sigma}{d\sigma} = -\Delta_q \pi_{\sigma^-}(1 - \pi_{\sigma^-}) < 0 \tag{C.4}
\]

Differentiating \( \Delta \pi_{\sigma} \) w.r.t \( \pi_{\sigma^-} \) yields:

\[
\frac{\partial \Delta \pi_{\sigma}}{\partial \pi_{\sigma^-}} = \frac{\Delta_q \left[ (1 - 2\pi_{\sigma^-})(\pi_{\sigma^-}\Delta_q + \Delta_q) - \pi_{\sigma^-}(1 - \pi_{\sigma^-})\Delta_q \right]}{(\pi_{\sigma^-}\Delta_q + \Delta_q)^2} \tag{C.5}
\]
I can now use the previous two relations to take the total derivative of $\pi_\sigma$ w.r.t $\sigma$:

\[
\frac{d\pi_\sigma}{d\sigma} = \left(1 + \frac{\partial \Delta \pi_\sigma}{\partial \pi_\sigma^-}\right) \frac{d\pi_\sigma^-}{d\sigma} = \frac{\lambda_q (\Delta_q + \Delta_\lambda)}{(\pi_\sigma^- - \Delta_q + \Delta_\lambda)^2} \frac{d\pi_\sigma^-}{d\sigma} \leq 0
\]  

(C.6)

To conclude the verification of $w$ being decreasing in $\sigma$ note that for $\sigma \in [0, \tau]$

\[
\frac{dw}{d\sigma} = -re^{-r\sigma}w(\pi_\sigma) + e^{-r\sigma}w'(\pi_\sigma)\frac{d\pi_\sigma}{d\sigma} \leq 0,
\]  

(C.7)

because of (C.6) and the fact that $w$ is assumed increasing on $[0, 1]$ and hence is a.e differentiable with positive derivative.

**Increasing in $\pi$.** To show that $w$ is increasing in $\pi$, I must hold $\sigma$ fixed, thus it remains to show $w(\pi_\sigma, \tau)$ is increasing in the initial belief $\pi$. It is pretty straightforward to show that the posterior belief at any time, for Poissonian environment that we have, is increasing in the initial belief, hence the proof readily follows from the increasing property of $w$. ||