The Impact of Connectivity on the Production and Diffusion of Knowledge

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Outline

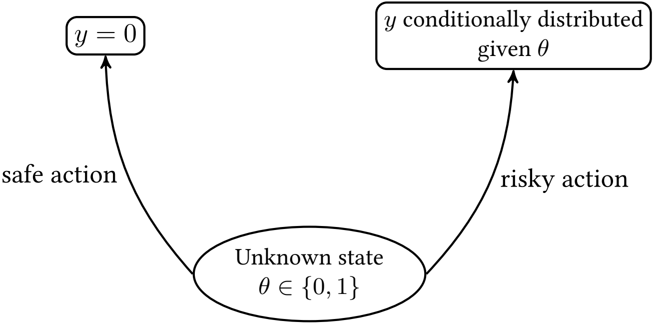
1. Introduction

2. Two-player Economy

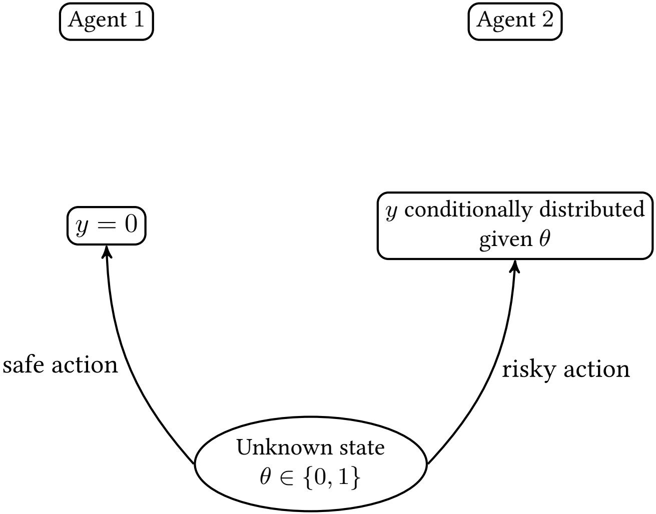
3. Equilibria in Large Economy

4. Social Surplus

A Bandit Framework



A Bandit Framework



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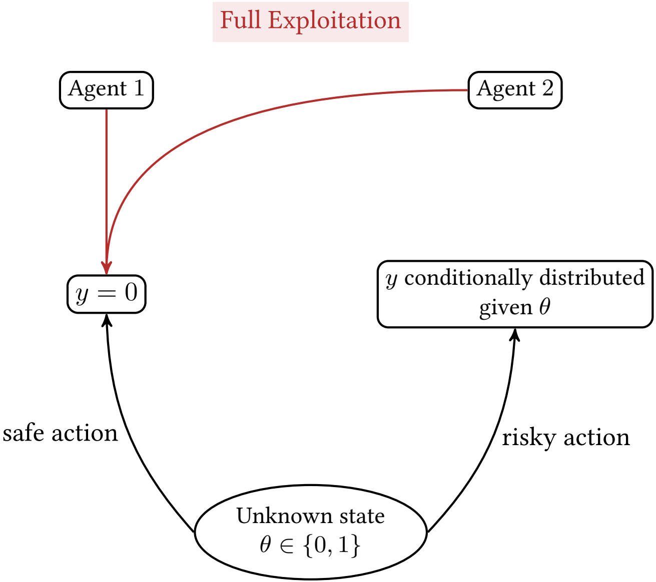

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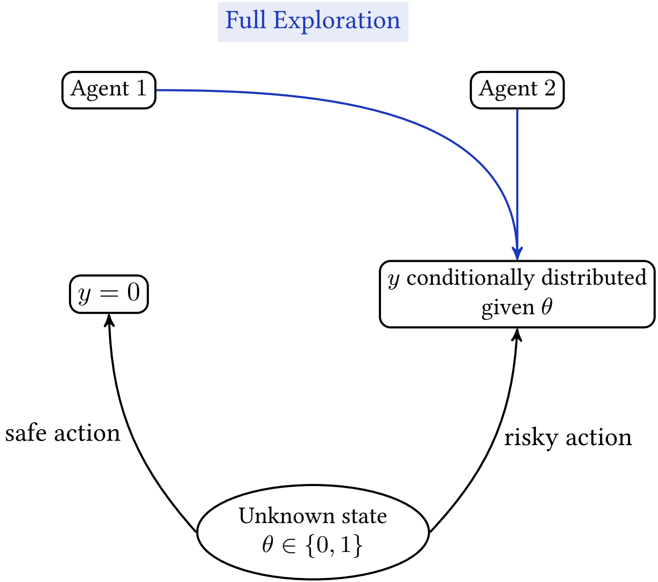

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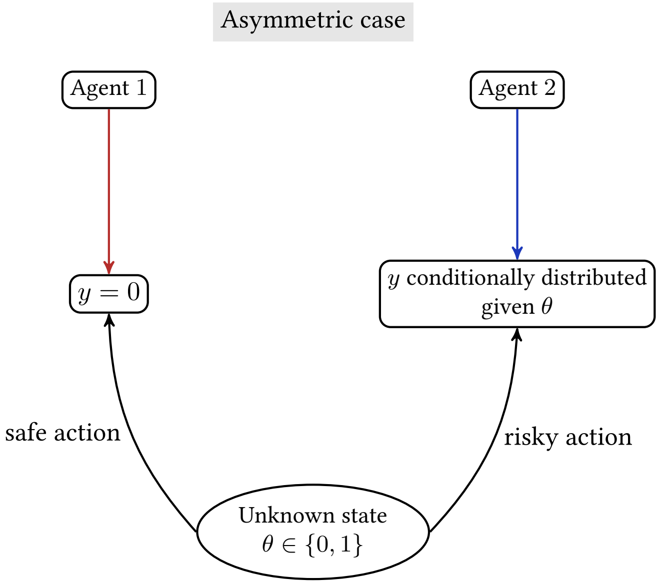

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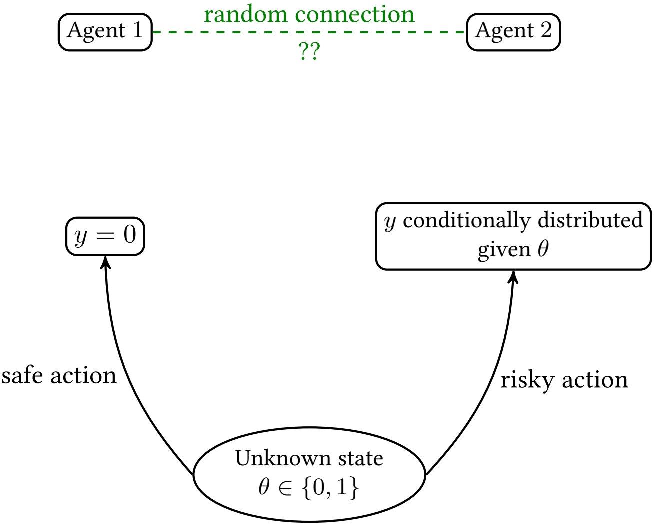


Main Contributions

- Multiple agents are connected via their social ties.
- Not fully connected like Bolton and Harris (1999) and Keller, Rady, and Cripps (2005), and not fully isolated like Gittins (1979).
- Our contribution: Intermediate case, where agents are connected to each other with some probability.
- Pros and cons of connections: *reducing* ex ante exploration incentives (because of free riding) and *increasing* ex post information sharing.

Equilibrium social surplus dependence on the degree of connections? Not necessarily increasing

Two-period Model

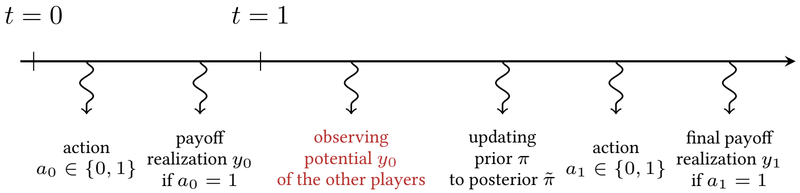
- Hidden payoff relevant state of the world $\theta \in \{0, 1\}$. Symmetric initial belief $\pi = \mathsf{P}(\theta = 1)$.

- Each agent faces a binary choice in both periods: $a = 0$ exploiting the safe arm; $a = 1$ exploring the risky arm.

- Payoff to the safe arm $= 0$.

- If agent selects the risky arm ($a = 1$) its payoff $y \in \{-\alpha, 1\}$ is drawn independently from

$$\mathsf{P}(y = 1 | \theta = 1) = \beta \text{ and } \mathsf{P}(y = 1 | \theta = 0) = 0.$$



Outline

Starting with Two-player economy

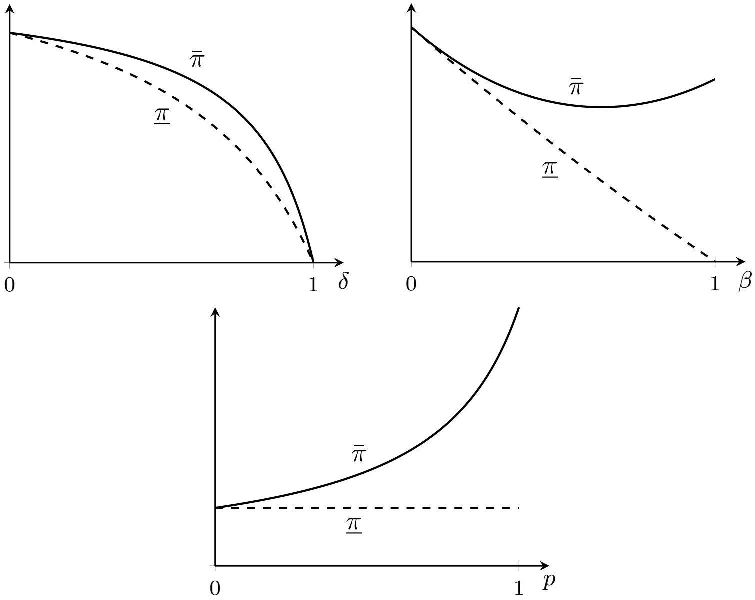
- Suppose each player gets to observe the outcome of the other player's first period experimentation with probability $p > 0$.
- Time discount factor $= \delta \in (0, 1) \rightarrow$ unlike other social learning models in networks, where agents are *myopic*: Bala and Goyal (1998), Gale and Kariv (2003) and Sadler (2020).

Proposition 1: Two-player equilibrium

There exists two thresholds $\underline{\pi} < \bar{\pi}$ such that the exploitation equilibrium appears only on $[0, \underline{\pi}]$, and the exploration equilibrium appears only on $(\bar{\pi}, 1]$. Closed form expressions for the cutoffs are

$$\underline{\pi} = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta}, \quad \bar{\pi} = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta(1-p\beta)}.$$

Some Comparative Statics



Social Optimum

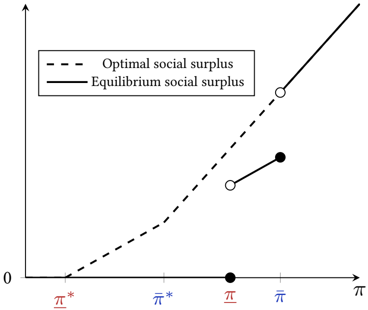
Proposition 2: Two-player optimum

The socially optimal outcome is for both players to exploit the safe arm whenever $\pi \leq \underline{\pi}^*$, and to jointly explore the risky arm on $\pi \geq \bar{\pi}^*$, where

$$\underline{\pi}^* = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta(1+p)}, \quad \bar{\pi}^* = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta\left(1+p(1-2\beta)\right)}.$$

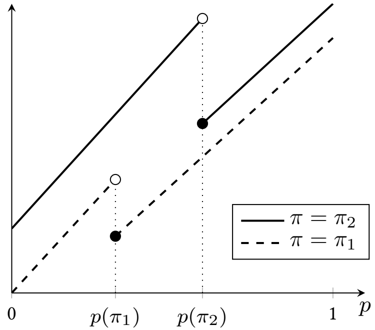
Under-exploration

Over-exploitation



Two-Player Equilibrium Social Surplus

As a function of connection probability p, it is **not** always increasing:



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Connection Graph

- $n \in \mathbb{N}$ agents in the economy.
- Each player in the second period observes the exploration outcome of a randomly selected group of individuals $\rightsquigarrow$ random variable M
- Two important cases:
  - **Local** observability: signals of the *immediate* neighbors
  - **Global** observability: signals of the *connected component* $\mathcal{C}$
- Some notation $\rightarrow$ let P_k and E_k refer to the distribution of M when $k \in \{0, 1, \ldots, n\}$ players *explore* in the first the period. Also denote

$$q_k(m) := \mathsf{P}_k(M = m) \text{ and } Q_k(m) := \mathsf{P}_k(M \leq m).$$

Equilibrium Characterization

Theorem 3: Equilibrium number of explorers

The equilibrium in which k players explore, where $0 < k < n$, exists only when

$$\frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta\mathsf{E}_{k-1}\left[(1-\beta)^M\right]} < \pi \leq \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta\mathsf{E}_{k}\left[(1-\beta)^M\right]}.$$

Full exploration (i.e. $k = n$) appears when

$$\pi > \bar{\pi} := \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta\mathsf{E}_{n-1}\left[(1-\beta)^M\right]},$$

and full exploitation (i.e. $k = 0$) appears on

$$\pi \leq \underline{\pi} := \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta}.$$

Large-n Limit of Equilibria with Local Observability

- Suppose every two agents are connected to each other with probability $p = \lambda/n$.
- As $n \to \infty$, the *full* exploration threshold converges to:

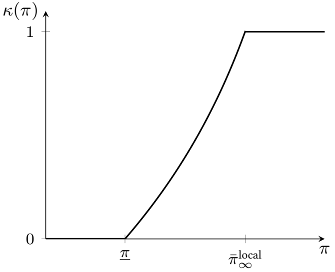
$$\bar{\pi}_{\infty}^{\text{local}} := \lim_{n\to\infty} \bar{\pi}_{n}^{\text{local}} = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta)+\delta\beta e^{-\lambda\beta}}$$

Proposition 4: Limiting fraction of explorers

Let $k_n(\pi)$ be the equilibrium number of exploring agents in an economy of n individuals with local connections, then

$$\lim_{n\to\infty} \frac{k_n(\pi)}{n} = \kappa(\pi)$$

$$:= \begin{cases} 0 & \pi \le \underline{\pi} \\ \frac{1}{\lambda\beta}\log\frac{\delta\pi\beta}{(1-\delta)(\alpha(1-\pi)-\pi)} & \underline{\pi} < \pi < \bar{\pi}_{\infty}^{\text{local}} \\ 1 & \pi \ge \bar{\pi}_{\infty}^{\text{local}} \end{cases}$$



Large-n Limit of Equilibria with Global Observability

Proposition 5: Limit of exploration threshold

Let $p = \lambda/n$ be the pairwise connection probability, and T be the total progenies of a Branching process with $\text{Poisson}(\lambda)$ offspring distribution, then

(i) $|\mathcal{C}|$ converges in distribution to T, where $\mathsf{P}(T = k) = e^{-\lambda k}\frac{(\lambda k)^{k-1}}{k!}$, and

(ii) as $n \to \infty$:

$$\bar{\pi}_{\infty}^{\text{global}} := \lim_{n\to\infty} \bar{\pi}_{n}^{\text{global}} = \frac{\alpha(1-\delta)}{(1+\alpha)(1-\delta) + \delta\beta\mathsf{E}\left[(1-\beta)^{T-1}\right]}.$$

Rapid Tightening of the Exploration Region

- The exploration threshold in the global regime is closely connected to the MGF of Borel's distribution $\rightarrow$ Lambert's W function
- For small β: $\bar{\pi}_\infty^{\text{global}}$ rapidly rises as λ goes from just below 1 to just above 1.

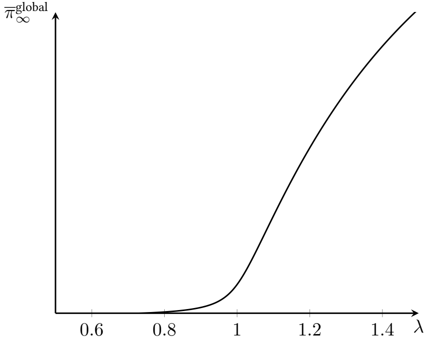


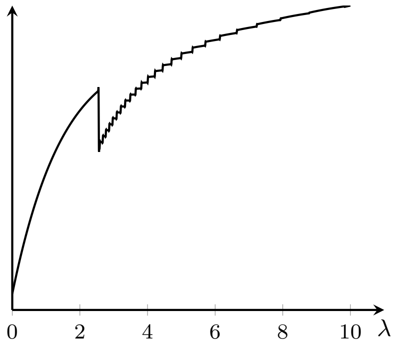
Figure 3: Criticality of $\lambda = 1$

Outline

Finite-n Equilibrium Social Surplus

Proposition 6: Finite-n average equilibrium social surplus

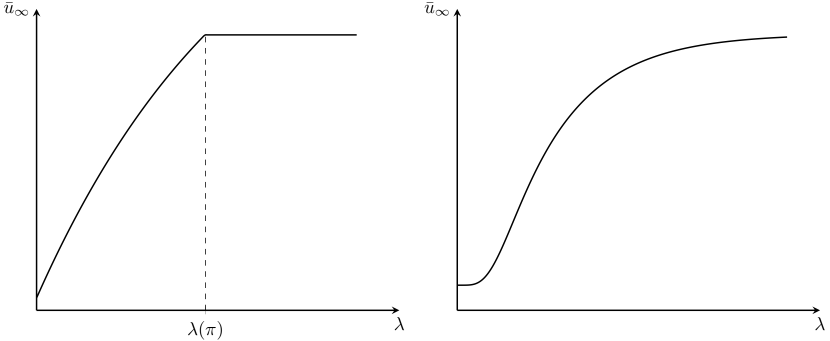
The equilibrium social surplus falls discontinuously on every λ where the economy undergoes an equilibrium regime change.



Large-n Limit of Equilibrium Social Surplus

Let k_n be the equilibrium number of exploring agents. Define

$$\bar{u}_\infty := \lim_{n\to\infty} \frac{u_{k_n,n}(\pi,\lambda)}{n}.$$



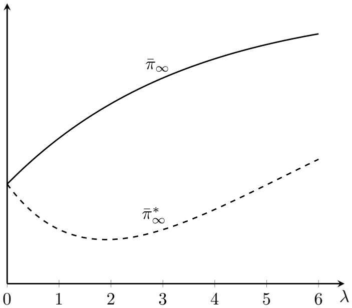
(a) $\underline{\pi} < \pi < \frac{\alpha}{1+\alpha}$

(b) $\pi \geq \frac{\alpha}{1+\alpha}$

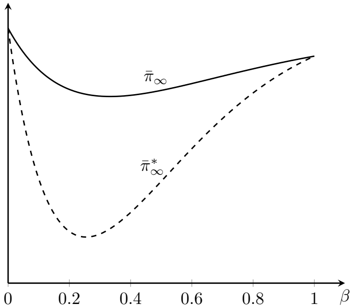
Social Optimum in Local Economies

Theorem 7: Social optimum

The socially optimal outcome is full exploitation iff $\pi \leq \underline{\pi}^*$, and full exploration iff $\pi \geq \overline{\pi}^*$. Furthermore, on $[0, \underline{\pi}^*]$ the social surplus is decreasing in k ($\Delta u_k \leq 0$), and on $[\overline{\pi}^*, 1]$ it is increasing in k ($\Delta u_k \geq 0$).



(a) Effect of λ



(b) Effect of β

Asymptotic Complementarity

The social welfare function features **asymptotic complementarity** between k and π, when for every $k \in \mathbb{N}$ and $\pi' < \pi''$ in $[0, 1]$:

$$\liminf_{n\to\infty} \min_{0\leq k<n} \left\{ \left(u_{k+1}(\pi'') - u_k(\pi'')\right) - \left(u_{k+1}(\pi') - u_k(\pi')\right) \right\} \geq 0$$

Proposition 8: Sufficient condition for asymptotic complementarity

For sufficiently small δ (specifically $\delta \leq \frac{1}{\lambda+2}$), or equivalently sufficiently sparse connections, the social welfare function features asymptotic complementarity.

Conclusion

- We characterize the equilibrium behavior in Bandits with random connections among agents.
- The limit of equilibria are found when $n \to \infty$ in economies with local and global observability of signals.
- Because of two involving forces, namely information sharing and free riding, the equilibrium social surplus is not always increasing in connections.
- We find sufficient condition for the existence of complementarity between the size of exploring group and the initial belief.